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## THE MATHEMATICAL GAZETTE.

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### THE GEOMETRIC INTERPRETATION OF HOMO- GRAPHIC EQUATIONS AND THEIR APPLICATION TO LOCI AND ENVELOPES.\*

#### A CHAPTER ON THE THEORY OF CROSS-RATIO.

BY REV. JOHN J. MILNE.

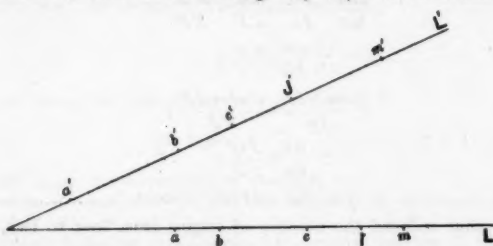
If two lines are divided in such a way that any point on one corresponds to one and only one point on the other, the lines are said to be divided homographically, and any relation between the segments which gives the variable point on one corresponding uniquely to a variable point on the other is a homographic equation.

These equations are of two types:

$$Axx' + Bx + Cx' + D = 0, \dots\dots\dots(\alpha)$$

$$Bx + Cx' + D = 0. \dots\dots\dots(\beta)$$

( $\alpha$ ), which contains the term  $xx'$ , may be said to be of the second order, and ( $\beta$ ), in which the term  $xx'$  is wanting, may be said to be of the first order.



The object of the present paper is  
Firstly, to show how these equations arise, and what are the meanings  
of the coefficients  $A$ ,  $B$ ,  $C$ ,  $D$ , and  
Secondly, to show how they can be applied to geometrical loci and envelopes.

\* A treatise on the *Cross-Ratio Geometry applied to the Point Line and Conic*, by the present writer, is published by the Pitt Press.

Let  $abc$  be three given points on a line  $L$ , and  $a'b'c'$  three given points on a line  $L'$ , and let us suppose that  $a'$  corresponds to  $a$ ,  $b'$  to  $b$ , and  $c'$  to  $c$ . Then, if  $m$  is any variable point on  $L$ , the positions of the six given points will enable us to determine the variable point  $m'$  on  $L'$  which corresponds to the variable point  $m$  on  $L$  by means of the homographic relation

$$(abcm) = (a'b'c'm').$$

From this, by expansion, we have

$$\frac{ac}{am} : \frac{bc}{bm} = \frac{a'c'}{a'm'} : \frac{b'c'}{b'm'},$$

or

$$\frac{am}{bm} = \left( \frac{ac}{bc} : \frac{a'c'}{b'c'} \right) \frac{a'm'}{b'm'} \\ = \mu \cdot \frac{a'm'}{b'm'}, \dots\dots\dots(1) \quad \text{where} \quad \mu = \frac{ac}{bc} : \frac{a'c'}{b'c'}.$$

This is a homographic equation in which there are four origins, viz.  $a$ ,  $b$  on  $L$ , and  $a'$ ,  $b'$  on  $L'$ , and  $\mu$  is a constant which we shall find of great importance.

We will now find the point on each range corresponding to the point at infinity on the other. Calling these points  $I$ ,  $J'$ , we have  $(abcI) = (a'b'c'\infty')$ .

$$\therefore \frac{ac}{aI} : \frac{bc}{bI} = \frac{a'c'}{a'\infty'} : \frac{b'c'}{b'\infty'} = \frac{a'c'}{b'c'};$$

$$\therefore \frac{aI}{bI} = \frac{ac}{bc} : \frac{a'c'}{b'c'} = \mu;$$

$$\therefore aI = \mu bI = \mu(aI - ab);$$

$$\therefore aI(\mu - 1) = \mu ab;$$

$$\therefore aI = \frac{\mu}{\mu - 1} ab, \quad bI = \frac{1}{\mu - 1} ab.$$

Similarly

$$a'J' = \frac{1}{\mu - 1} b'a', \quad b'J' = \frac{\mu}{\mu - 1} b'a'.$$

We can now obtain our second homographic equation.

For, since  $(aIm\infty) = (a'\infty'm'J')$ ,

$$\therefore \frac{am}{a\infty} : \frac{Im}{I\infty} = \frac{a'm'}{a'J'} : \frac{\infty'm'}{\infty'J'};$$

$$\therefore \frac{am}{Im} = \frac{a'm'}{a'J'};$$

$$\therefore \frac{am - Im}{Im} = \frac{a'm' - a'J'}{a'J'};$$

$$\therefore \frac{aI}{Im} = \frac{J'm'}{a'J'};$$

$$\therefore Im \cdot J'm' = aI \cdot a'J' = \text{const.} \dots\dots\dots(2)$$

We have thus reduced the number of origins from four to two, viz. the points  $I$ ,  $J'$ .

This equation is often useful, but as the two origins  $I$ ,  $J'$  are special points, we require one of a more general form.

Suppose we wish to obtain an equation in which the origins are any two non-corresponding points  $a$ ,  $b'$ .

Putting  $Im = am - aI$  and  $J'm' = b'm' - b'J'$  in (2), we have

$$(am - aI)(b'm' - b'J') = Ia \cdot J'a',$$

which reduces to

$$am \cdot b'm' - b'J' \cdot am - aI \cdot b'm' + aI \cdot b'a' = 0, \dots\dots\dots(3)$$

and putting

$$aI = -\frac{\mu}{1-\mu}ab \text{ and } b'J' = -\frac{\mu}{1-\mu}b'a',$$

this becomes

$$(1-\mu)am \cdot b'm' + \mu b'a' \cdot am + \mu ab \cdot b'm' - \mu ab \cdot b'a' = 0. \dots\dots\dots(3')$$

If we wish to make the origins two corresponding points  $a, a'$ , we must write

$$Im = am - aI \text{ and } J'm' = a'm' - a'J',$$

and we obtain

$$am \cdot a'm' - a'J' \cdot am - aI \cdot a'm' = 0, \dots\dots\dots(4)$$

$$(1-\mu)am \cdot a'm' + b'a' \cdot am + \mu ab \cdot a'm' = 0. \dots\dots\dots(4')$$

Comparing (3) and (4) with (a), we see that in homographic equations of the type  $Axx' + Bx + Cx' + D = 0$ :

(i) If the origins are non-corresponding points,  $D$  cannot equal zero.

If  $C=0$ , the origin for  $x$  is at  $I$ .

If  $B=0$ , the origin for  $x'$  is at  $J'$ .

(ii) If the origins are corresponding points,  $D=0$ , but neither  $B$  nor  $C$  can vanish.

Hence, if the origins are a pair of corresponding points, there is no absolute term, and conversely, if the absolute term is wanting the origins are corresponding points.

Now, if the ranges are in perspective, the points at their intersection correspond. Consequently, when the intersection of the ranges is the common origin, if the homographic equation has no absolute term the ranges are in perspective.

#### PROPORTIONAL SECTION.

In equations 1-4,  $\mu$  may have any numerical value except zero or infinity. Let us consider the case where  $\mu=1$ . Equation (1) becomes

$$\frac{am}{a'm'} = \frac{bm}{b'm'} = \frac{am-bm}{a'm'-b'm'} = \frac{ab}{a'b'} = \text{const.} \dots\dots\dots(5)$$

and we have the case of proportional section.

Hence, if two lines are divided proportionally, they are divided homographically, and the points at infinity correspond.

If we put  $\mu=1$ ,

$$(3) \text{ becomes } \frac{am}{ab} + \frac{b'm'}{b'a'} = 1, \dots\dots\dots(6)$$

the origins being the non-corresponding points  $a, b'$ .

$$(4) \text{ becomes } \frac{am}{ab} = \frac{a'm'}{a'b'}, \dots\dots\dots(5')$$

the origins being the corresponding points  $a, a'$ .

Comparing (5) and (6) with ( $\beta$ ), we see that in equations of the type  $Bx + Cx' + D = 0$ :

(iii) If the origins are non-corresponding points, neither  $B, C$ , nor  $D$  can vanish.

(iv) If the origins are corresponding points,  $D=0$ .

And in all cases of ( $\beta$ )  $I$  and  $J'$  are at an infinite distance, and the points at infinity are corresponding points.

A treatise on Proportional Section by Apollonius, in two books containing 181 Propositions, was discovered in Arabic in the Bodleian Library, and translated into Latin by Halley in 1706.

## ILLUSTRATIONS.

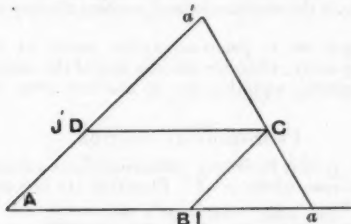
As practical illustrations of my subject, which is of very wide application, I have selected a few examples which are chiefly concerned with showing that straight lines which move subject to certain conditions pass through fixed points, and that points which satisfy certain laws lie on fixed straight lines.

In any given question we can generally ascertain whether the homographic equation is of the second or first order, by determining by inspection whether the points  $I, J'$  are at a finite or an infinite distance.

The properties on which such examples depend are that, if a pair of corresponding points coincide at the intersection of two ranges, the ranges are in perspective, and the lines joining pairs of corresponding points all pass through the same point; and if the ranges are not in perspective, the lines envelop a conic touching the ranges.

Also, if the line joining the centres of two pencils is a *common ray*, the pencils are in perspective, and the intersections of pairs of corresponding rays lie on a straight line; and if the pencils are not in perspective, the intersections lie on a conic passing through the centres.

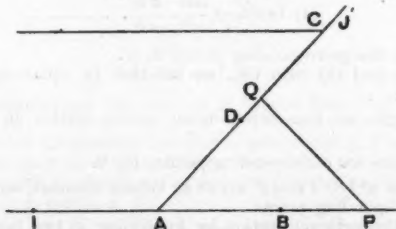
1. Through the angle  $C$  of a parallelogram  $ABCD$  is drawn a straight line meeting the two sides  $AB, AD$  in  $a, a'$ . Prove that the rectangle  $Ba, Da'$  is constant.



The ranges  $(a), (a')$  are sections of the same pencil, centre  $C$ , and are therefore homographic. When  $a'$  is at infinity,  $a$  is at  $B$ , which is therefore the position of  $I$ ; and when  $a$  is at infinity,  $a'$  is at  $D$ , which is therefore  $J'$ . Hence, by equation (2),  $Ba \cdot Da'$  is constant. Also, if we make  $a, a'$  coincide at  $A$ , the constant is seen to be the product  $AB \cdot AD$ .

Suppose we wish to write down the homographic equation of the ranges  $(a), (a')$ . They are evidently in perspective, since  $a$  and  $a'$  coincide at  $A$ , and, by (4), the equation is

$$xx' - ADx - ABx' = 0.$$



2.  $AB, AC$  are two given straight lines, on which  $B$  and  $C$  are fixed and  $P, Q$  movable points on them, such that  $AP : AB = AQ : QC$ .

Prove that the straight line  $PQ$  passes through a fixed point.

Let  $AP = x$ ,  $AQ = x'$ .

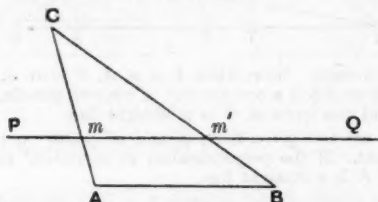
Then the given condition is  $x : AB = x' : AC - x'$ ;

i.e.

$$xx' - AC \cdot x + AB \cdot x' = 0.$$

Therefore, by equation (4), as there is no absolute term,  $A$  is a common point,  $I$  is on  $BA$  produced through  $A$ , so that  $BA = AI$ ,  $J'$  is at  $C$ , the ranges are in perspective, and  $PQ$  in its different positions always passes through a fixed point.

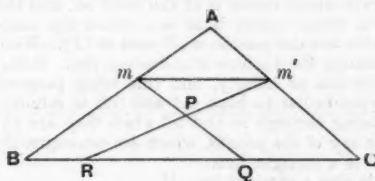
3. Given the base  $AB$  of a triangle  $ABC$ , and the length of the segment  $mm'$  which the sides intercept on a fixed line  $PQ$  parallel to  $AB$ , show that the locus of the vertex  $C$  is a straight line.



The ranges  $(m)$ ,  $(m')$  are identical, and therefore homographic and of the first order. Hence the pencils  $A(m)$  and  $B(m')$  are homographic. Therefore the locus of  $C$  is either a straight line or a conic, according as the pencils  $A(m)$  and  $B(m')$  have or have not a common ray. Now, when  $m$  is at infinity, so also is  $m'$ . Therefore  $AB$  is a common ray, the pencils are in perspective, and the locus of  $C$  is a straight line.

In the same question, if  $a$  and  $a'$  are fixed points on  $PQ$ , and the ratio of the segments  $am$ ,  $a'm'$  is given, the locus of  $C$  is a straight line. For  $am = k \cdot a'm'$ . Therefore, by equation (5),  $PQ$  is divided proportionally at  $m$ ,  $m'$ , etc.

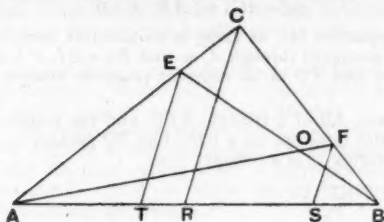
4.  $Q$ ,  $R$  are fixed points on  $BC$ , the base of a fixed triangle  $ABC$ . A line  $mm'$ , parallel to the base, meets the sides  $AB$ ,  $AC$  in  $m$ ,  $m'$ . If  $Qm$ ,  $Rm'$  meet in  $P$ , the locus of  $P$  is a straight line.



Since  $\frac{Am}{Am'} = \frac{\sin C}{\sin B} = \text{constant}$ , by equation (5) the ranges  $(m)$ ,  $(m')$  are homographic and of the first order, the pencils  $Q(m)$ ,  $R(m')$  are homographic. Also, when  $m$  is at  $B$ ,  $m'$  is at  $C$ ; therefore  $QR$  is a common ray, the pencils are in perspective, and the locus of  $P$  is a straight line.

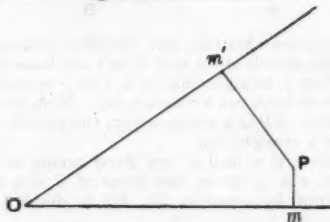
5. If, on  $AB$  the base of a triangle  $ABC$ , we take two variable points  $S$ ,  $T$ , such that  $AT$  is in a constant ratio to  $BS$ , and draw  $ET$  and  $FS$  parallel to a fixed line  $CR$ , meeting  $AC$  in  $E$  and  $BC$  in  $F$ ; then, if  $AF$ ,  $BE$  meet in  $O$ , the locus of  $O$  is a straight line.

By equation (5) the ranges ( $T$ ) and ( $S$ ) are homographic, as are also ( $T$ ) and ( $E$ ), and ( $S$ ) and ( $F$ ). Therefore ( $E$ ) and ( $F$ ) are homographic, and therefore also the pencils  $A(F)$  and  $B(E)$ . Therefore the locus of  $O$  is either



a straight line or a conic. Now, when  $E$  is at  $A$ ,  $T$  is at  $A$ ,  $S$  is at  $B$ , and  $F$  is at  $B$ . Therefore  $AB$  is a common ray of the two pencils, which are thus in perspective, and the locus of  $O$  is a straight line.

6.  $OA$  and  $OB$  are two given lines,  $m$  and  $m'$  points on them, such that  $Om + Om' = \text{constant}$ . If the perpendiculars at  $m$  and  $m'$  meet in  $P$ , show that the locus of  $P$  is a straight line.



The relation between  $m$  and  $m'$  being of the form  $x + x' = \text{constant}$ , the ranges ( $m$ ) and ( $m'$ ), by equation (6), are homographic, and being similar,  $m$  and  $m'$  are at infinity together. The series of perpendiculars ( $Pm$ ) constitute a pencil of parallel rays whose centre is at the point  $\infty$ , and the perpendiculars ( $Pm'$ ) form a pencil whose centre is at  $\infty'$ . Since the ranges ( $m$ ), ( $m'$ ) are homographic, so also are the pencils  $\infty(P)$  and  $\infty'(P)$ . We have to consider whether the line joining the vertices is a common ray. Now, the line joining their vertices is the line at infinity, and this being perpendicular to every straight line is perpendicular to both  $OA$  and  $OB$  at infinity, and is the ray of each pencil passing through  $m$  and  $m'$  when they are at infinity, and is therefore a common ray of the pencils, which are consequently in perspective, and the locus of  $P$  is a straight line.

The locus of  $P$  is also a straight line, if

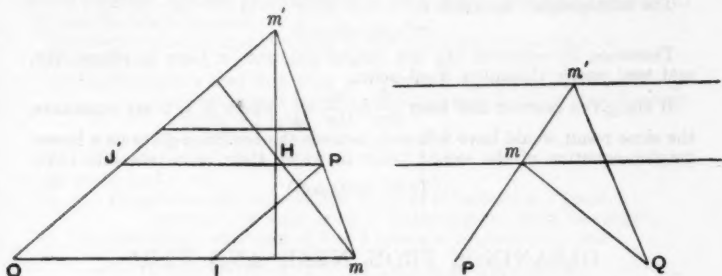
- (1)  $k \cdot Om + l \cdot Om' = n$ , where  $k, l, n$  are constants.
- (2)  $mm'$  moves parallel to a given direction.

7. Find the locus of the orthocentre of a triangle two of whose sides are given in position, and whose base passes through a fixed point  $R$ .

Let  $O$  be the vertex of the triangle,  $Pmm'$  any position of the base, and through  $P$  draw lines parallel to the sides meeting  $Om$  in  $I$  and  $Om'$  in  $J'$ . Let  $H$  be the orthocentre of the triangle  $Omm'$ . Then the ranges ( $m$ ), ( $m'$ ) are homographic of the second order, since  $I$  and  $J'$  are obviously at a finite distance, and  $O$  is a common point. Hence the homographic equation is

$$xx' - OJ' \cdot x - OI \cdot x' = 0.$$

As in Ex. 6, the series of perpendiculars  $mH$ ,  $m'H$  form two homographic pencils  $\infty(H)$ ,  $\infty'(H)$ , but the line joining their centres, i.e. the line at infinity, does not cut the ranges in corresponding points because their homographic equation is of the second order, and therefore the points at infinity do not correspond. Therefore the locus of  $H$  is a conic, passing through two points at infinity, and therefore a hyperbola.



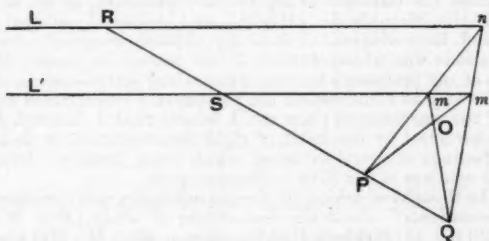
8. Through a given point  $P$  a straight line is drawn meeting two fixed parallel straight lines in  $m$ ,  $m'$ , and through  $m$ ,  $m'$  straight lines are drawn in given directions meeting in  $Q$ . Prove that the locus of  $Q$  is a straight line.

The ranges  $(m)$ ,  $(m')$ , being sections of the same pencil by two parallel lines, are homographic. Since  $m$  and  $m'$  are at infinity together, and  $mm'$  always passes through the fixed point  $P$ , the ranges  $(m)$ ,  $(m')$  are in perspective, and their equation is of the form

$$Bx + Cx' = 0.$$

The series of rays  $Qm$ ,  $Qm'$  constitute two pencils  $\infty(m)$ ,  $\infty'(m')$ , and the line joining their centres, i.e. the line at infinity, contains a pair of corresponding values of  $m$  and  $m'$ , since the homographic equation is of the first order. Therefore the pencils are in perspective, and the locus of  $Q$  is a straight line.

9. A point  $n$  moves along a line  $L$ , and is joined to two fixed points  $P$ ,  $Q$ , and the lines  $Pn$ ,  $Qn$  meet a fixed line  $L'$  parallel to  $L$  in  $m$ ,  $m'$ . Prove that  $O$ , the intersection of  $Pm'$  and  $Qm$  describes a straight line.



The ranges  $(m)$ ,  $(n)$  are homographic, as are also  $(m')$ ,  $(n)$ , and therefore also  $(m)$  and  $(m')$ , and the pencils  $P(m')$  and  $Q(m)$ . Therefore the locus of  $O$  is either a straight line or a conic.

Let  $QP$  produced meet  $L$  in  $R$  and  $L'$  in  $S$ . Then when  $n$  is at  $R$ ,  $m$  and  $m'$  coincide at  $S$ , and  $QP$  is a common ray. Hence the locus of  $O$  is a straight line.

10. Given the vertical angle of a triangle, and the sum of the reciprocals of the sides containing it, show that the base will always pass through a fixed point.

In the fig. of § 7, let  $O$  be the vertex,  $mm'$  any position of the base. Then

$$\frac{1}{Om} + \frac{1}{Om'} = \frac{1}{k}.$$

The homographic equation is

$$xx' - kx - kx' = 0.$$

Therefore, by equation (4), the ranges ( $m$ ) and ( $m'$ ) are in perspective, and  $mm'$  passes through a fixed point.

If the given relation had been  $\frac{l}{Om} + \frac{n}{Om'} = \frac{1}{k}$ , where  $k, l, n$  are constants, the same result would have followed, because the condition gives us a homographic equation of the second order in which there is no absolute term.

(To be continued.)

## GLEANINGS FROM NEAR AND FAR.\*

"IL N'Y A PAS DE MAUVAIS DOCUMENTS."

Taine to J. E. C. Bodley.

1. "In my childhood I was praised for the readiness with which I could multiply and divide, by memory alone, two sums of several figures; such praise encouraged my growing talent; and had I persevered in this line of application, I might have acquired some fame in mathematical studies."—*Memoirs of my Life and Writings*, Edward Gibbon, p. 31 (1900, ed. by Birkbeck Hill).

"The best translators from the Greek, for instance, I find to be very poor arithmeticians."—*Decline and Fall*, v. 407 n. (Bury, 1897).

"Arithmetic is an excellent touchstone to try the amplifications of passion and rhetoric."—*Loc. cit.* vi. 405.

"From a blind idea of the usefulness of such abstract science, my father had been desirous, and even pressing, that I should devote some time to the mathematics; nor could I refuse to comply with so reasonable a wish. During two winters I attended the private lectures of Monsieur De Traytorrens, who explained the elements of algebra and geometry, as far as the conic sections of the Marquis de l'Hôpital, and appeared satisfied with my diligence and improvement. But as my childish propensity for numbers and calculations was wholly extinct, I was content to receive the passive impression of my professor's lectures without any active exercise of my own powers. As soon as I understood the principles, I relinquished for ever the pursuit of the mathematics; nor can I lament that I desisted, before my mind was hardened by the habit of rigid demonstration, so destructive of the finer feelings of moral evidence, which must, however, determine the actions and opinions of our lives."—*Memoirs*, p. 95.

In 1762 he thought of taking up the pursuit again, and consulted "a very able mathematician" about the best course of study (*Misc. Works*, Lord Sheffield, 1814, ii. 44; Birkbeck Hill, *loc. cit.* n. p. 95). Mr. Hill also refers to J. S. Mill (*Autobiography*, 1873, p. 19): "The boasted influence of mathematical studies is nothing to it [the school logic], for in mathematical processes none of the real difficulties of correct ratiocination occur."

\* It is in many ways an advantage for each article to open at the head of a page. The spaces left at the foot of an article will be utilised for these "Gleanings," the items being numbered for convenience of reference.

THE PRINCIPLES OF PROBABILITY AND  
 APPROXIMATIONS IN ARITHMETIC.

BY W. HOPE-JONES, B.A.

(Continued from p. 8.)

3. If  $n$  quantities, which are known to the nearest whole number, are added together, find the probability that their sum so obtained is correct to the nearest whole number.

Let the probability that the error is between  $x$  and  $x+dx$  be  $p\,dx$ .

If  $n=2$ , and  $x>0$ ,

$p\,dx$  = Probability that first error is between  $x-\frac{1}{2}$  and  $\frac{1}{2}$ ,  
 and that second error is within a given  $dx$  of its possible range  
 from  $-\frac{1}{2}$  to  $\frac{1}{2}$ ,  
 $= (1-x)dx$ ;  $\therefore p=1-x$ .

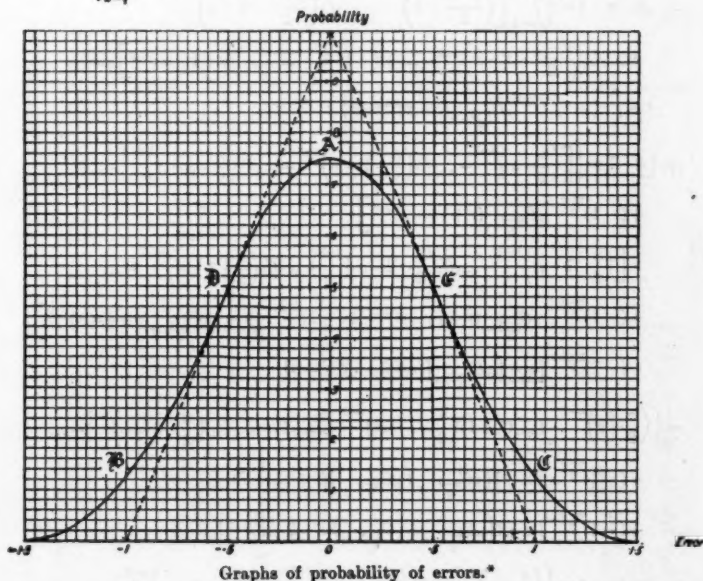
If  $x<0$ ,  $p=1+x$ .

If  $n=3$ , and  $x>\frac{1}{2}$ ,

$p\,dx$  = Probability that sum of first 2 errors is between  $x-\frac{1}{2}$  and 1,  
 and that the third error is within a given  $dx$  of its range ;

$\therefore p$  = Probability that sum of first 2 errors is between  $x-\frac{1}{2}$  and 1

$$= \int_{x-\frac{1}{2}}^1 (1-x)dx = (1\frac{1}{2}-x)^2/2. \text{ If } x < -\frac{1}{2}, p = (1\frac{1}{2}+x)^2/2.$$



\* The squared paper makes any inscription indistinct. The dotted lines are  $p=1\pm x$ . The reader may insert

at A (in red ink)  $p = \{(1\frac{1}{2} \pm x)^2 - 3(\frac{1}{2} \pm x)^2\} / 2$ ,

at B  $p = (1\frac{1}{2} + x)^2 / 2$ ,

at C  $p = (1\frac{1}{2} - x)^2 / 2$ ,

at D, E equation changes here if  $n=3$ .

If  $\frac{1}{2} > x > -\frac{1}{2}$ ,

$p$  = Probability that sum of first 2 errors is between  $x \pm \frac{1}{2}$

$$= \int_{x-\frac{1}{2}}^0 (1+x)dx + \int_0^{x+\frac{1}{2}} (1-x)dx = \{(1\frac{1}{2}-x)^2 - 3(\frac{1}{2}-x)^2\} / 2.$$

The equation giving  $p$  in terms of  $x$  changes at every integer if  $n$  is even, at every integer  $+\frac{1}{2}$  if  $n$  is odd.

Let  ${}_n p_r dx$  be the probability that the error is between  $x$  and  $x+dx$ , when  $x$  is between  $r \pm \frac{1}{2}$ ,  $r$  being an integer if  $n$  is odd, an integer  $+\frac{1}{2}$  if  $n$  is even.

In the cases so far worked out,

$$\begin{aligned} {}_n p_r \times \underline{n-1} &= \left(\frac{n}{2}-x\right)^{n-1} - {}_n C_1 \left(\frac{n}{2}-1-x\right)^{n-1} + {}_n C_2 \left(\frac{n}{2}-2-x\right)^{n-1} - \dots \\ &\quad + (-1)^{\frac{n-1}{2}-r} {}_n C_{\frac{n-1}{2}-r} (r+\frac{1}{2}-x)^{n-1}. \end{aligned}$$

By induction it can be proved that this is so in all cases.

Proof by induction (assuming the formula for  ${}_{n-1} p_{r-1}$  and  ${}_{n-1} p_{r+1}$ ).

$${}_n p_r = \int_{x-\frac{1}{2}}^r {}_{n-1} p_{r-1} dx + \int_r^{x+\frac{1}{2}} {}_{n-1} p_{r+1} dx = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} {}_{n-1} p_{r-1} dx + \int_{x-\frac{1}{2}}^r {}_{n-1} p_{r-1} - {}_{n-1} p_{r+1} dx; \quad (5)$$

$$\begin{aligned} \therefore {}_n p_r \underline{n-2} &= \left\{ \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \left[ \left(\frac{n-1}{2}-x\right)^{n-2} - {}_{n-1} C_1 \left(\frac{n-1}{2}-1-x\right)^{n-2} \right. \right. \\ &\quad \left. \left. + {}_{n-1} C_2 \left(\frac{n-1}{2}-2-x\right)^{n-2} - \dots + (-1)^{\frac{n-3}{2}-r} {}_{n-1} C_{\frac{n-3}{2}-r} (r+1-x)^{n-2} \right] dx \right. \\ &\quad \left. + \int_{x-\frac{1}{2}}^r (-1)^{\frac{n-1}{2}-r} {}_{n-1} C_{\frac{n-1}{2}-r} (r-x)^{n-2} dx \right\}; \\ {}_n p_r \underline{n-1} &= \left\{ \left[ \left(\frac{n}{2}-x\right)^{n-1} - {}_{n-1} C_1 \left(\frac{n}{2}-1-x\right)^{n-1} + {}_{n-1} C_2 \left(\frac{n}{2}-2-x\right)^{n-1} - \dots \right. \right. \\ &\quad \left. \left. + (-1)^{\frac{n-3}{2}-r} {}_{n-1} C_{\frac{n-3}{2}-r} (r+1\frac{1}{2}-x)^{n-1} \right] \right. \\ &\quad \left. - \left[ \left(\frac{n}{2}-1-x\right)^{n-1} - {}_{n-1} C_1 \left(\frac{n}{2}-2-x\right)^{n-1} + {}_{n-1} C_2 \left(\frac{n}{2}-3-x\right)^{n-1} - \dots \right. \right. \\ &\quad \left. \left. + (-1)^{\frac{n-5}{2}-r} {}_{n-1} C_{\frac{n-5}{2}-r} (r+1\frac{1}{2}-x)^{n-1} + (-1)^{\frac{n-3}{2}-r} \underline{n-1} {}_{n-1} C_{\frac{n-3}{2}-r} (r+\frac{1}{2}-x)^{n-1} \right] \right. \\ &\quad \left. + (-1)^{\frac{n-1}{2}-r} \underline{n-1} {}_{n-1} C_{\frac{n-1}{2}-r} [(r+\frac{1}{2}-x)^{n-1} - 0] \right\} \\ &= \left\{ \left(\frac{n}{2}-x\right)^{n-1} - ({}_{n-1} C_1 + 1) \left(\frac{n}{2}-1-x\right)^{n-1} + ({}_{n-1} C_1 + {}_{n-1} C_2) \left(\frac{n}{2}-2-x\right)^{n-1} + \dots \right. \\ &\quad \left. + (-1)^{\frac{n-3}{2}-r} \underline{n-1} \left( {}_{n-1} C_{\frac{n-3}{2}-r} + {}_{n-1} C_{\frac{n-5}{2}-r} \right) (r+1\frac{1}{2}-x)^{n-1} \right. \\ &\quad \left. + (-1)^{\frac{n-1}{2}-r} \underline{n-1} \left( {}_{n-1} C_{\frac{n-1}{2}-r} + {}_{n-1} C_{\frac{n-3}{2}-r} \right) (r+\frac{1}{2}-x)^{n-1} \right\}; \\ \therefore {}_n p_r \underline{n-1} &= \left\{ \left(\frac{n}{2}-x\right)^{n-1} - {}_n C_1 \left(\frac{n}{2}-1-x\right)^{n-1} + {}_n C_2 \left(\frac{n}{2}-2-x\right)^{n-1} + \dots \right. \\ &\quad \left. + (-1)^{\frac{n-1}{2}-r} {}_n C_{\frac{n-1}{2}-r} (r+\frac{1}{2}-x)^{n-1} \right\} \dots \dots \dots (6) \end{aligned}$$

*Note.*—This series continues as far as the last term in which the quantity which has the index  $(n-1)$  is positive.

TABLE OF VALUES OF  $u_r^p$ .

	$r=0$	$r=\frac{1}{2}$	$r=1$		
$n=1$	$(\frac{1}{2}-x)^0/0$	$(1-x)^0/1$	$(1\frac{1}{2}-x)^0/2$		
$n=2$	$\{(1\frac{1}{2}-x)^2-3(\frac{1}{2}-x)^2\}/2$	$\{(2-x)^2-4(1-x)^2\}/3$	$\{(2\frac{1}{2}-x)^2-5(1\frac{1}{2}-x)^2\}/4$		
$n=3$	$\{(2\frac{1}{2}-x)^4-5(1\frac{1}{2}-x)^4+10(\frac{1}{2}-x)^4\}/4$	$\{(3-x)^4-6(2-x)^4+15(1-x)^4\}/5$	$\{(3\frac{1}{2}-x)^4-7(2\frac{1}{2}-x)^4+21(1\frac{1}{2}-x)^4\}/6$		
$n=4$					
$n=5$					
$n=6$					
$n=7$	$\{(3\frac{1}{2}-x)^6-7(2\frac{1}{2}-x)^6+21(1\frac{1}{2}-x)^6-35(\frac{1}{2}-x)^6\}/6$	$\{(4-x)^6-8(3-x)^6+28(2-x)^6-56(1-x)^6\}/7$			
$n=8$					
	$r=1\frac{1}{2}$	$r=2$	$r=2\frac{1}{2}$	$r=3$	$r=3\frac{1}{2}$
$n=1$					
$n=2$					
$n=3$	$(2-x)^3/3$	$(2\frac{1}{2}-x)^4/4$	$(3-x)^5/5$	$(3\frac{1}{2}-x)^6/6$	$(4-x)^7/7$
$n=4$					
$n=5$	$\{(3-x)^3-6(2-x)^3\}/5$	$\{(3\frac{1}{2}-x)^6-7(2\frac{1}{2}-x)^6\}/6$	$\{(4-x)^7-8(3-x)^7+28(2-x)^7\}/7$		
$n=6$					
$n=7$					
$n=8$					

To find the probability that the error is between  $\pm \frac{1}{2}$ .

Since  ${}_n p_r = \int_{x-\frac{1}{2}}^x {}_{n-1} p_{r-1} dx + \int_x^{x+\frac{1}{2}} {}_{n-1} p_r dx$ , by (5).

$\therefore$  by putting  $r=0$ , if  $n$  is odd,

$${}_n p_0 = \int_{x-\frac{1}{2}}^0 {}_{n-1} p_{-1} dx + \int_0^{x+\frac{1}{2}} {}_{n-1} p_0 dx;$$

$\therefore$  value of  ${}_n p_0$  when  $x=0$  is  $\int_{-\frac{1}{2}}^0 {}_{n-1} p_{-1} dx + \int_0^{\frac{1}{2}} {}_{n-1} p_0 dx$

= probability that the error is between  $\pm \frac{1}{2}$  when  $(n-1)$  quantities are added.

If  $n$  is even, put  $r = \frac{1}{2}$ .

$$\text{Then } {}_n p_{\frac{1}{2}} = \int_{x-\frac{1}{2}}^{\frac{1}{2}} {}_{n-1} p_0 dx + \int_{\frac{1}{2}}^{x+\frac{1}{2}} {}_{n-1} p_{\frac{1}{2}} dx;$$

$\therefore$  value of  ${}_n p_{\frac{1}{2}}$  when  $x=0$  is  $\int_{-\frac{1}{2}}^{\frac{1}{2}} {}_{n-1} p_0 dx$

= probability that the error is between  $\pm \frac{1}{2}$  when  $(n-1)$  quantities are added.

$\therefore$  if  $n$  quantities are added, the probability that the sum so obtained is correct to the nearest whole number is the value, when  $x=0$ , of  ${}_{n+1} p_0$  or  ${}_n p_{\frac{1}{2}}$

$$= \frac{1}{n!} \left\{ \left( \frac{n}{2} + \frac{1}{2} \right)^n - {}_{n+1} C_1 \left( \frac{n}{2} - \frac{1}{2} \right)^n + {}_{n+1} C_2 \left( \frac{n}{2} - 1 \right)^n - \dots \text{as far as the term containing } \left( \frac{1}{2} \right)^n \text{ or } 1^n \right\} \dots \dots \dots (7)$$

Call this probability  $P(n)$ .

TABLE OF VALUES OF  $P(n)$  AND  $\sqrt{6/(n+1)\pi}$ .

$n$	$P(n)$	$\sqrt{6/(n+1)\pi}$	$\sqrt{6/(n+1)\pi} - P(n)$
1	1	·9772	- ·0228
2	·75	·7979	+ ·0479
3	·6667	·6910	+ ·0243
4	·5990	·6180	+ ·0190
5	·55	·5642	+ ·0142
6	·5110	·5223	+ ·0113
7	·4794	·4886	+ ·0092
8	·4529	·4607	+ ·0078
9	·4304	·4370	+ ·0066
10	·4110	·4167	+ ·0057

Mean square of error

$$\begin{aligned}
 &= \left\{ \int_{-\frac{n}{2}}^{\frac{n}{2}} {}_n p_{\frac{n}{2}-x} dx + \int_{-\frac{n}{2}}^{\frac{n}{2}-1} {}_n p_{\frac{n}{2}-1-x} dx + \dots + \int_{-\frac{n}{2}}^{1-\frac{n}{2}} {}_n p_{\frac{1}{2}-x} dx \right\} x^2 dx \\
 &= \frac{1}{n-1} \left\{ \int_{-\frac{n}{2}}^{\frac{n}{2}} x^2 \left( \frac{n}{2} - x \right)^{n-1} dx - \int_{-\frac{n}{2}}^{\frac{n}{2}-1} x^2 {}_n C_1 \left( \frac{n}{2} - 1 - x \right)^{n-1} dx \right. \\
 &\quad \left. + \int_{-\frac{n}{2}}^{\frac{n}{2}-2} x^2 {}_n C_2 \left( \frac{n}{2} - 2 - x \right)^{n-1} dx - \dots + (-1)^{n-1} \int_{-\frac{n}{2}}^{1-\frac{n}{2}} x^2 {}_n C_{n-1} \left( 1 - \frac{n}{2} - x \right)^{n-1} dx \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{n-1} \left\{ \int_0^n y^{n-1} \left( \frac{n}{2} - y \right)^2 dy - {}_nC_1 \int_0^{n-1} y^{n-1} \left( \frac{n}{2} - 1 - y \right)^2 \right. \\
 &\quad \left. + {}_nC_2 \int_0^{n-2} y^{n-1} \left( \frac{n}{2} - 2 - y \right)^2 - \dots + (-1)^{n-1} \int_0^1 y^{n-1} \left( 1 - \frac{n}{2} - y \right)^2 dy \right\} \\
 &= \frac{1}{n-1} \sum_{t=0}^{n-1} (-1)^t {}_nC_t \int_0^{n-t} y^{n-1} \left( \frac{n}{2} - t - y \right)^2 dy \\
 &= \frac{1}{n-1} \sum_0^{n-1} (-1)^t {}_nC_t \int_0^{n-t} \left\{ \left( \frac{n}{2} - t \right)^2 y^{n-1} - (n-2t)y^n + y^{n+1} \right\} dy \\
 &= \frac{1}{n-1} \sum_0^{n-1} (-1)^t {}_nC_t \left[ \left\{ \left( \frac{n}{2} - t \right)^2 (n-t)^n \right\} / n - \frac{1}{2} (n-2t)(n-t)^{n+1} / (n+1) \right. \\
 &\quad \left. + (n-t)^{n+2} / (n+2) \right] \\
 &= \frac{1}{n+2} \sum_0^{n-1} (-1)^t {}_nC_t (n-t)^n \left[ (n+1)(n+2) \left( \frac{n}{2} - t \right)^2 - n(n+2)(n-2t)(n-t) \right. \\
 &\quad \left. + n(n+1)(n-t)^2 \right].
 \end{aligned}$$

Let  $n-t=s$ ;

$$\begin{aligned}
 \therefore \text{mean (error)}^2 &= \frac{1}{n+2} \sum_1^n (-1)^{n-s} {}_nC_s s^n \left[ (n+1)(n+2) \left( \frac{n}{2} - s \right)^2 + ns(n+2)(n-2s) \right. \\
 &\quad \left. + ns^2(n+1) \right] \\
 &= \frac{1}{n+2} \sum_1^n (-1)^{n-s} {}_nC_s s^n \left[ \frac{n^2}{4} (n+1)(n+2) - ns(n+2) + 2s^2 \right].
 \end{aligned}$$

Now it is easy to show that

$$\sum_1^n (-1)^{n-s} {}_nC_s s^n = n, \quad \sum_1^n (-1)^{n-s} {}_nC_s s^{n+1} = \frac{n}{2} (n+1),$$

$$\text{and} \quad \sum_1^n (-1)^{n-s} {}_nC_s s^{n+2} = \frac{n(3n+1)}{24} (n+2).$$

$$\therefore \text{mean (error)}^2 \times (n+2)$$

$$= \left\{ \left[ n \frac{n^2}{4} (n+1)(n+2) - \frac{n}{2} (n+1) n(n+2) + \frac{2n(3n+1)}{24} (n+2) \right] \right\}$$

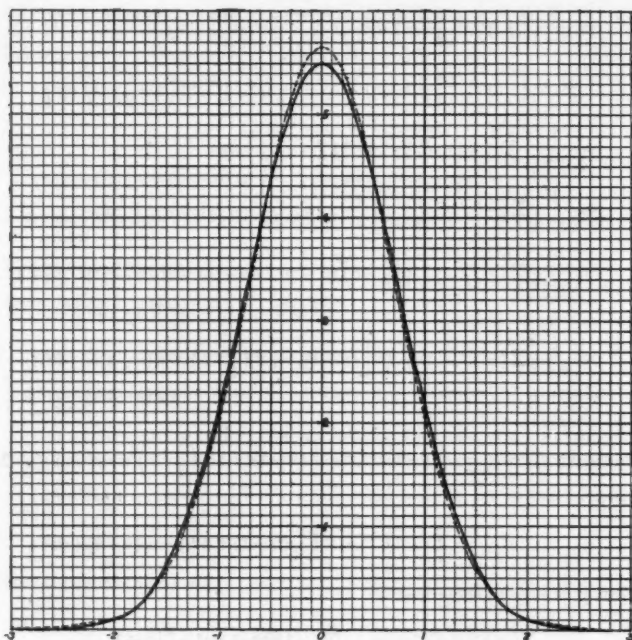
$$\therefore \text{mean (error)}^2 = \frac{n}{12}, \dots \dots \dots (8)$$

The curve  $y = \sqrt{6/(n\pi)} e^{-6x^2/n}$  has the same mean (error)<sup>2</sup> and area (namely 1) as  $y=p$ , which approximates closely to it when  $n$  is fairly large. Even when  $n$  is only 6, the curves are almost indistinguishable when  $x^2 > \frac{1}{4}$ . (See graphs.)

$$\text{Taking } p \text{ as } \sqrt{6/(n\pi)} e^{-6x^2/n}, \quad P(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{6/(n\pi)} e^{-6x^2/n} dx$$

$$\begin{aligned}
 &= \sqrt{6/(n+1)\pi} - \text{terms containing } n^{-2} \text{ and higher negative powers of } n \\
 &= \sqrt{6/(n+1)\pi} \text{ (approx.)} \dots \dots \dots (9)
 \end{aligned}$$

If  $p$  is taken as  $\sqrt{6/n\pi} e^{-6x^2/n}$ , probable error =  $0.6745\sqrt{n/12}$ ; but in practice  $0.6745\sqrt{(n+\frac{1}{2})/12}$  is a better approximation.



..... Graph of  $y = e^{-x^2/\pi}$ , the value of  $\sqrt{6/n\pi} e^{-6x^2/n}$  when  $n=6$ .

—— Graph of  $y = \frac{1}{5} \left\{ \begin{array}{l} (3-x)^5 \text{ from } x=0 \text{ to } x=3 \\ -6(2-x)^5 \text{ from } x=0 \text{ to } x=2 \\ +15(1-x)^5 \text{ from } x=0 \text{ to } x=1 \end{array} \right\}$   
 (or the corresponding quantity for negative values of  $x$ ),  
 the value of  $p$  when  $n=6$ .

TABLE OF VALUES OF PROBABLE ERROR,  $\cdot 6745\sqrt{n/12}$  AND  $\cdot 6745\sqrt{(n+\frac{1}{4})/12}$

$n$	Probable Error.	$\cdot 6745\sqrt{n/12}$	$\cdot 6745\sqrt{(n+\frac{1}{4})/12}$
1	$\cdot 25$	$\cdot 1947$	$\cdot 2177$
2	$\cdot 2929$	$\cdot 2754$	$\cdot 2921$
3	$\cdot 3529$	$\cdot 3372$	$\cdot 3510$
4	$\cdot 4027$	$\cdot 3894$	$\cdot 4014$
12	$\cdot 6817$	$\cdot 6745$	$\cdot 6815$

4.  $n$  decimals of £1, each correct to 3 places, are added.

To find the probability that the sum so obtained is correct to the nearest penny.

Let  $\mathcal{E}x/10^3$  = the error in the sum, and let  $c$  be the probability that the sum is incorrect, that is, that  $\mathcal{E} \frac{\text{an integer} + \frac{1}{2}}{240}$  comes between the approximate value  $\mathcal{E}a/10^3$  and the exact value  $\mathcal{E}(a-x)/10^3$ ,  $a$  being an integer.

$2a/10^3 = 24a$  pence; and  $24a$  can in no case differ from an integer +  $\frac{1}{2}$  by less than .02.

$\therefore c=0$  if  $(24x \sim 0) < .02$ , that is, if  $(x \sim 0) < \frac{1}{12}$ .

If  $(x \sim 0)$  is between  $\frac{1}{12}$  and  $\frac{1}{6}$ , then

$c$  = the probability that  $x$  is positive, and  $a$  of the form  $25m+23$   
 + " "  $x$  is negative, and  $a$  " "  $25m+2$   
 $= \frac{1}{25}$ .

If  $(x \sim 0)$  is between  $\frac{1}{6}$  and  $\frac{1}{3}$ , then for the approximate answer to be incorrect,  $a$  must be of the form  $25m+(23 \text{ or } 19)$  if  $x > 0$ ,  $25m+(2 \text{ or } 6)$  if  $x < 0$ .  $\therefore c = \frac{2}{25}$ .

Range of  $(x \sim 0)$ .

Corresponding value of  $c$ .

$< \frac{1}{12}$	0
$\frac{1 \text{ to } 3}{12}$	$\frac{1}{25}$
$\frac{3 \text{ to } 5}{12}$	$\frac{2}{25}$
$\frac{5 \text{ to } 7}{12}$	$\frac{3}{25}$
$\frac{7 \text{ to } 9}{12}$	$\frac{4}{25}$
$\frac{9 \text{ to } 11}{12}$	$\frac{5}{25}$
$\frac{11 \text{ to } 13}{12}$	$\frac{6}{25}$
$\frac{13 \text{ to } 15}{12}$	$\frac{7}{25}$
$\frac{15 \text{ to } 17}{12}$	$\frac{8}{25}$
$\frac{17 \text{ to } 19}{12}$	$\frac{9}{25}$
$\frac{19 \text{ to } 21}{12}$	$\frac{10}{25}$
$\frac{21 \text{ to } 23}{12}$	$\frac{11}{25}$
$\frac{23 \text{ to } 25}{12}$	$\frac{12}{25}$
$> \frac{25}{12}$	1

By (6) the frequency of any error  $x$  is

$$\frac{1}{n-1} \left\{ \left( \frac{n}{2} - x \right)^{n-1} - {}_nC_1 \left( \frac{n}{2} - 1 - x \right)^{n-1} + {}_nC_2 \left( \frac{n}{2} - 2 - x \right)^{n-1} - \dots \text{as far as the last term in which the quantity which has the index } (n-1) \text{ is positive} \right\}.$$

If only positive values of  $x$  are considered, the value so obtained for  $c$  will be half the right value.

$$\therefore \frac{c}{2} (n-1) = \left\{ \frac{1}{25} \int_{\frac{1}{12}}^{\frac{1}{6}} + \frac{2}{25} \int_{\frac{1}{6}}^{\frac{1}{3}} + \frac{3}{25} \int_{\frac{1}{3}}^{\frac{1}{2}} + \dots + \frac{3n}{25} \int_{\frac{(n-1)}{12}}^{\frac{n}{12}} \right\} \left\{ \begin{array}{l} \left( \frac{n}{2} - x \right)^{n-1} \text{ from } x=0 \text{ to } x=\frac{n}{2} \\ - {}_nC_1 \left( \frac{n}{2} - 1 - x \right)^{n-1} \text{ from } 0 \text{ to } \frac{n}{2} - 1 \\ + {}_nC_2 \left( \frac{n}{2} - 2 - x \right)^{n-1} \text{ from } 0 \text{ to } \frac{n}{2} - 2 \\ \dots \end{array} \right\} dx;$$

$$\therefore \frac{25c}{2} (n-1) = \left\{ \int_{\frac{1}{12}}^{\frac{n}{12}} + \int_{\frac{1}{6}}^{\frac{n}{6}} + \int_{\frac{1}{3}}^{\frac{n}{3}} + \dots + \int_{\frac{(n-1)}{12}}^{\frac{n}{12}} \right\} \left( \frac{n}{2} - x \right)^{n-1} dx \\ - {}_nC_1 \left\{ \int_{\frac{1}{12}}^{\frac{n}{12}-1} + \int_{\frac{1}{6}}^{\frac{n}{6}-1} + \int_{\frac{1}{3}}^{\frac{n}{3}-1} + \dots + \int_{\frac{(n-1)}{12}}^{\frac{n}{12}-1} \right\} \left( \frac{n}{2} - 1 - x \right)^{n-1} dx \\ + \dots;$$



If the decimals are correct to 4 places instead of 3, the formula corresponding to (11) is :

$$\frac{125c \cdot 6^n \cdot |n}{2} = (\text{if } n \text{ is even}), \text{ the sum of first } \frac{3n}{2} \text{ (odd numbers)}^n$$

$$- {}_nC_1 \quad " \quad " \quad \frac{3n}{2} - 3 \quad "$$

$$+ {}_nC_2 \quad " \quad " \quad \frac{3n}{2} - 6 \quad "$$

$$- \dots \dots \dots ,$$

or (if  $n$  is odd), the sum of first  $(3n-1)/2$  (even numbers)<sup>n</sup>

$$- {}_nC_1 \quad " \quad " \quad (3n-1)/2 - 3 \quad "$$

$$+ {}_nC_2 \quad " \quad " \quad (3n-1)/2 - 6 \quad "$$

$$- \dots \dots \dots (14)$$

In this case only the last 125 odd or even numbers are to be counted in any group containing more than 125 ; but this does not happen until  $n=84$ , and reduces  $c$  only by 2 per cent. when  $n=5000$ .

TABLE OF VALUES OF  $c$ , WHEN THE ORIGINAL DECIMALS ARE CORRECT TO 4 PLACES.

$n$	$c$	$\sqrt{.000031n}$
1	.0053	.0056
2	.0078	.0079
3	.0096	.0096
4	.0111	.0111
5	.0124	.0124
6	.0135	.0136

These results give us our reason for teaching our pupils to work to 4 places of decimals of £1 in Compound Interest Questions.

(To be continued.)

2. "I see now what great advantage there is in giving two years to mathematics ; everything becomes clearer and easier."—*The Life of Pasteur*, René Vallery-Radot, 1906, p 19.

"By a touching exchange of parts, the child to whom his father had taught the alphabet now became his father's teacher : but with what respect. . . The father would often sit up late at night over rules of grammar and mathematical problems, preparing answers to send to his boy in Paris."—*Loc. cit.* p. 30.

3. "Mathematicians of to-day are like moles, each in his own little burrow finding his way about apparently by a sixth sense and turning up symmetrical rubbish heaps—a useful servant but a very bad master. We have no mathematicians of the type of Maxwell or Rankine, with views wide enough to co-ordinate all the known facts of physics. The amateur to-day is plainly told that there is no place for him ; and that engineers have ceased to discuss these questions is shown by their non-contribution to this discussion."—F. G. Edwards, "Electron, Quantal and Relativity Theories," *Engineering*, Dec. 8, 1916.

## SOME INCIDENTAL WRITINGS BY DE MORGAN.

(Continued from p. 83.)

[A glance at the Index of the *D.N.B.* shows that Arthur Hopton was an astrologer and mathematician—of Clement's Inn—friend of Selden—published prognostications for the years 1607-1614, wrote *Baculum Geodacticum*, 1610, and similar works, but died in 1614.

Clement's Inn was in Hopton's day a subordinate Inn of Chancery attached to the Inner Temple, which so far corresponds with A. E. B.'s statement above.

But it does not seem that Hopton was at Oxford. On the other hand (Sir) Arthur Hopton (1588 ?-1650), uncle of the Baron Hopton who fought for Charles in the Civil War, was an Oxford man, and not a member of the Temple. The diplomatic life of the future ambassador hardly fits in with the publication of almanacs, though 1615 is a date which would in his case allow of the publication of the duodecimo in question. Was the *Concordancie* a posthumous production by the astrologer ? Here the invaluable S. comes to our aid :

Hopton, Arthur, astrologer. A CONCORDANCY OF YEARES; containing a new, easie, and most exact COMPUTATION OF TIME according to the English Account; also the Use of the English and Roman KALENDER, with briefe Notes, Rules, and Tables, as well Mathematicall and Legal, as Vulgar, for each priuate Mans occasion, newly composed, digested and augmented. 12mo, black letter, 1616, for the Company of Stationers.

S. describes it as "one of the many astrological and prophetic almanacs of which the Stationers' Company long exercised an undignified monopoly." The dedication is as stated by A. E. B. Included is "a briefe discourse of the natural causes of watery meteors, as snow, hail, raine, etc.; divers signes to prognosticate what weather is towards; certain predictions of the eather in every Moneth," etc. We are also told that this work is mentioned by the commentators on Shakespeare, Allibone being the authority. Another astrologer, Dr. Simon Forman, helps to fix the date of Macbeth, for he went to see the play.<sup>1</sup> But for the moment I cannot find Hopton's name in this connection.

S. gives another of the rare publications of this "distinguished mathematician":

SPECULUM TOPOGRAPHICUM: or the TOPOGRAPHICALL GLASSE, containing the use of the Topographicall Glasse, Theodelitus, Plaine Table and Circumferenter, with many Rules of Geometry, Astronomy, Topography perspective, and Hydrography, sm. 4to, partly black letter, 1611, N.O. for Simon Water-son. This, says S., is mistakenly attributed by Anthony & Wood to the Ambassador above mentioned. The *D.N.B.* says that "it contains many good practical rules in geometry, measurement of distances, heights, sun's altitude and parallax, and a table for calculating annuities." On p. 3, adds S., is a map of the World, showing North and South America and "New Guinea."

*Elvington* is a misprint for *Elrlington*.]

## II.

1850. I. i. 170. Ptolemy of Alexandria.—There are several real or pretended translations of the *astrological* work—some certainly pretended—and Ptolemy's name is on many astrological title-pages which do not even pretend to translate. The Geography, as far as Britain is concerned, is said to be in Dr. Henry's *History of Great Britain*, 1788. Some works in harmonics appear in lists as translations or close imitations of Ptolemy, as John Keeble's, 1785; Francis Styles, *Phil. Trans.* vol. li. Various dissertations on minor pieces exist: but there is no English translation of the *Almagest*, &c., though it exists in French

<sup>1</sup> Forman's *Journal* (Halliwell's *Reprint* from Ashmolean MSS.), April 20, 1610.

(see Smith's *Biograph. Dict.* art. **PTOLEMY**). If an English reader wants to know Ptolemy's astronomical methods and hypotheses, nothing will suit him better than Narrien's *History of Astronomy*. M.

[The enquiry was as to any translated works of Ptolemy. A correspondent gave the five following :

*The Composit of Ptholomeus, Prynce of Astronomeye, translated out of Frenche into Englysshe*, 12mo, printed by Robert Wyer. n.d.

*The Bounding of Greece-Land, according to Ptolomeus : Englished out of the Greek*, by Thos. Wilson, 4to, London, 1570. (This is included in Wilson's Translation of Demosthenes' *Olynthiacs*.)

*The Geography of Ptolemy, so far as it relates to Britain ; in Greek and English, with observations by J. Horsley*, fol., London, 1732. (This forms part of the *Britannia Romana*.)

*Tetrabiblos, or Quadrapartite ; being Four Books, of the Influence of the Stars, newly translated from the Greek Paraphrase of Proclus ; with a Preface, Explanatory Notes, and an Appendix containing Extracts from the Almagest of Ptolemy, and the whole of his Colloquy, etc.*, by J. M. Ashmead. 8vo, London, 1822.

*Quadripartite ; or Four Books concerning the Influence of the Stars, faithfully rendered into English, from Leo Allatius ; with Notes, explaining the most difficult and obscure passages*, by John Whalley, 12mo, London, 1701 and 1786.

John Whalley was a well-known quack (1653-1724) and vendor of universal medicines—a self-styled Professor of Physic and Astrology. He published various astrological works, and claimed for the *Quadripartite* that "there is nothing in Astrology but what is there comprehended, nothing there comprehended, but the Quintessence and Divinity of Astrology."

Leo Allatius, or Leone Allacci (1586-1669), born in Chios, became librarian of the Vatican. He edited a number of translations from the classics.

De Morgan himself wrote the article in Smith's *Biog. Dict.* (1849), so the subject was fresh in his mind.

The definitive edition of Ptolemy's works is that by J. L. Heiberg : *Claudii Ptolemaei opera quae extant omnia. Almagest. Part I.* 1898. *Part II.* 1903. The following references may be useful : *Ptolemäus Claudius. Handbuch der Astronomie. Aus dem Griechischen überset.* . . . by Karl Manitius. 2 vols. 1912 → (Teubner).

**ΜΑΘΗΜΑΤΙΚΗ ΣΥΝΤΑΞΙΣ.** Composition Mathématique, traduite pour la première fois du grec en français, sur les MSS. originaux de la Bibliothèque Impériale (in vol. ii, du Roi) de Paris, par l'Abbé Nicolas B. Halma, et suivie des Notes de Jean Baptiste Joseph Delambre, 2 vols. roy. 4to, 1813-1816. S. says that only 400 copies were issued, and that the second volume, containing a medallion portrait of Louis XVIII., is most difficult to obtain.

*ὑποθέσεις τῶν πλανημένων. . . De Planetarum Hypothesisibus* was first published, London, 1620 (Gr. Lat.) by John Bainbridge, the first Savilian Professor of Astronomy at Oxford.

*Hypothèses et époques des planètes de Cl. Ptolémée et Hypotyposes de Proclus Diadochus*, by the Abbé Halma. Paris, 1820. This contains "summaries of the numerical data in the Syntaxis, commonly known as the *Almagest*, but with some deviations." As will be gathered from the title, Proclus wrote a commentary on it. Various minor translations of Ptolemy's works were published by Halma between 1820 and 1828.

There is an excellent chapter on the Ptolemaic System (pp. 191-206), in Dreyer's *History of the Planetary Systems from Thales to Kepler*, 1906 (Cam. Univ. Press) ; v. also R. Grant's *History of Physical Astronomy*, 1852, pp. 435-441, and a very readable chapter on Ptolemy in Sir Robert Ball's *Great Astronomers* (Iabister), 1895. On Ptolemy as an authority for the positions of the ancient constellations, v. Mr. Maunder's article, *Observatory*, Aug., 1913, p. 329.

The best edition of the Greek text of the *Geographia* is that by C. F. A. Nobbe, 3 vols. 1843 (Leipzig). "The best edition so far completed," say Messrs. Beasley and Bunbury, in the *Encyc. Brit.*, is that published in A. F. Didot's *Bibliotheca graecorum scriptorum (Claudii Ptolemaei geographia)*, 2 vols.

1893, 1901, Paris. They also refer to H. Bradley, *Ptolemy's Geography of the British Isles*, in *Archæologia*, xlviii. 1885, and to T. G. Rylands, *Geography of Ptolemy Elucidated*, 1893, Dublin.

The various editions of the *Almagest* have been collated and published by C. H. F. Peters and E. B. Knobel (Carnegie Institute of Washington), new edition, 1916.

*Nature*, July 5, 1917, refers to the "standard edition" of the geography by C. Mueller, continued by C. Th. Fischer, and "still incomplete." For Mr. L. O. T. Tudee's examination of the maps attached to various MSS. v. *Journal of Hellenic Studies*, 1917, vol. 37, part i.

Robert Henry, D.D. (1718-1790), Moderator of the General Assembly (1774), received a pension of £100 a year from the British Government for his *History of Great Britain, Written on a New Plan*. 5 vols. 1771-1785. The sixth volume was edited by M. Laing, and published after Henry's death in 1793. The work was the object of a savage attack by Dr. Gilbert Stuart, for an account of which v. D'Israeli's *Calamities of Authors*.

*Ptolemæus (Claudius) Harmonicorum Libri III., ex. Codd. MSS. XI., nunc primum Græce editus.* Johannes Wallis, S.T.D., R.S.S., recensuit, edidit, Versione et Notis illustravit, et Auctarium adiecit. Oxonii, e Theatro Sheldoniano, 1682, sm. 4to. The Auctarium is *De Veterum Harmonica ad hodiernam comparata*. This treatise on Music was afterwards reprinted with Porphyry's Commentary in the third volume of Wallis's *Works*, Oxford, 1699. Pythagorean speculations on numbers form the basis of the third book.

John Keeble (1711-1786) attended lectures by Pepusch and "fell under the spell of his admiration for Greek music." Keeble was afterwards appointed organist at St. George's, Hanover Square, and at Ranelagh Gardens.

*A Theory of Harmonics, or an Illustration of the Grecian Harmonica*, by John Keeble. London, 1784.

Part I dealt with the systems of Euclid, Aristoxenus, and Bacchius.

Part II treated of the doctrine of the Ratio, etc., and the Pythagorean numbers in Nicomachus.

This work is "full of ingenious ideas, and the diagrams of strings and ratios proved of some interest" (D.N.B.).

Narrien, John, F.R.S. *Historical Account of the Origin and Progress of Astronomy*, 1833, reprinted 1850, 8vo.

He also wrote:

*Analytical Geometry, with the Properties of Conic Sections, and Appendix on Descriptive Geometry*, 8vo, 1846.

*Practical Astronomy and Geodesy, including Projections of the Sphere and Spherical Trigonometry*, 8vo, 1845.

The "Historical Account" is probably the work alluded to by Allman in the *Encyc. Brit.* as Narrien's *History of Astronomy*.

Narrien (1782-1860) was an optician and writer on astronomy. He was appointed Professor of Mathematics at Sandhurst in 1820, and elected F.R.S. in 1840. It would be interesting to discover something of Narrien, the man, as Prof. Turner tells me that the obituary notice in the *Monthly Notices of the R.A.S.* is disappointing in its brevity. The D.N.B. states that he was the son of a stone-mason, and was born at Chertsey. He kept an optician's shop in Pall Mall. His talents were soon recognised, and procured him sufficient patronage for a nomination as teacher of mathematics on the staff of the Royal Military College, at Sandhurst. In 1858 the failure of his sight led to his resignation and he died, æt. 77, at Kensington.

[Of Francis Styles I have so far found nothing.]

### III.

I. i. 281. **Arabic Numerals.**—Your correspondent should consult Peacock's "History of Arithmetic" in the *Encyclopædia Metropolitana*; and, if he can get them, the notes to Chasles' *Aperçu Historique des Méthodes en Géométrie*, and various papers of Mr. Chasles, published in the *Comptes Rendus* of the French Institute. He may perhaps find some information in De Morgan's *Arithmetical Books*, particularly at p. 14. M.

# INDIANA STATE NORMAL LIBRARY

SOME INCIDENTAL WRITINGS BY DE MORGAN. 117

E. V. (p. 230) had asked for references to works or treatises giving information on the history of the Arabic numerals, their origin, and introduction into Europe.

[Note the use of Mr. for M. Chasles. The reference to the *Aperçu* is pp. 464-476 (3rd edit. 1889).

The passage in his *Arithmetical Books* to which De Morgan refers occurs in his notes on :

"Venice, fifteen-twenty-three. SACROBOSCO. "Algorismus domini Joh. de Sacro Busco noviter impressum." Quarto.

... Dr. Peacock (p. 416) thinks that this work is attributed to Sacrobosco without sufficient reason, and mentions it only as a manuscript. Mr. Halliwell reprinted it in the *Rara Mathematica*, evidently under the impression that it had never been printed. If it be the work of Sacrobosco, it establishes beyond doubt that he had the Arabic numerals and the method of local value : being nothing but a summary of rules for that arithmetic. The words *noviter impressum* are ambiguous : they may either imply a first impression or any succeeding one, though I have commonly found them used in the latter sense.

I should pause before I rejected as spurious a work which is attributed to Sacrobosco in many manuscript copies extant in different countries, which was printed under his name as early as 1523, and is often cited as his. Dr. Peacock lays stress on there being no mention of the Arabic system in his other works, those on the sphere and on the calendar. But against the presumption drawn from this circumstance, it may be urged that it was not likely he would introduce mention of a new system of arithmetic in works intended for common use, though he might write a separate work to explain that system. And he would have no motive for alluding to a method which his readers were not acquainted with. There is great probability that Sacrobosco was acquainted with the modern system, which all other evidence goes to show was introduced into Italy before his time.

It is, of course, very possible, or, looking at the progress of other things, most probable, that isolated individuals had obtained the Arabic notation from the East, or from Italy, practised it, and written in it, long before it obtained the smallest general currency.

But there is one circumstance which seems to me to lend more than presumption to a still wider supposition, namely, that Arabic notation and rules were known beyond the bounds of Italy, in the time of Sacrobosco, to more than a few isolated individuals. Except by importations from the East, it is impossible to say whence the philosophers of the thirteenth century could have got anything like a set of rules. They certainly had not got anything Greek, except from the Arabs : the system of Boethius does not give rules of computation. Moreover, the name *algorithm* was in later times so invariably connected with Arabic arithmetic, that the presumption is strong it was so from the beginning. Now Roger Bacon, the contemporary of Sacrobosco, not only used the adjective *algoristicus* several times, but recapitulates the names of a set of rules. The theologian, he says (Jebb, p. 138), should abound in the power of numbering, that he may know all the *algoristic* modes, not only for integers, but for fractions, to numerate, to add, to subtract, to mediate (divide by two), to multiply, to divide, and to extract roots. Now this is precisely the set of rules given in the treatise attributed to Sacrobosco, with one omission : this last treatise distinguishes *duplation* (multiplication by two) from other cases of multiplication, which Bacon does not ; and *progression*, which is, however, only a mixture of other rules.

There is a sentence of Bacon's contemporary, Matthew Paris, quoted by Mr. Hallam, in which he speaks of what can be done with Greek notation, as being what cannot be done either in Latin, or in algorithm, *vel in algorismo*. These words might easily be interpolated ; so instead of using them, as Mr. Hallam does, to give ground of presumption that Bacon had the Arabic notation, I should rather use what I have drawn from Bacon to strengthen the genuineness of the words of Paris ; and I think Mr. Hallam will be ready to do the same. . . .

Peacock's *Arithmetic*, 4to, pp. 154, 1826. "The first 114 pp. contain a valuable history of arithmetic."—S.

The *Encyc. Metropolitana* article was not republished as will be seen from the following footnote to De Morgan's *Arithmetical Books*, p. xvii. Of the articles in the *Encyclopaedia Metropolitana*, he says :

"It was once intended to publish these treatises separately. Nine years ago [writing in 1847] the proprietors of the *Encyclopaedia Metropolitana* so fully intended to publish separately, that they considered themselves aggrieved because I, who had written the *mathematical* article on probabilities, wrote a popular work with that word in the title-page, which they alleged, through their agent, was in effect a republication of the former work, &c. Not being able to get them either to litigation or arbitration, I was obliged to write a pamphlet to prove that the charge was frivolous. The pamphlet is unanswered, and all the treatises unpublished (separately) to this day. The latter, I regret, for the sake of science. It is a great pity that Sir John Herschel's treatises on light and sound, Dr. Peacock's arithmetic, Mr. Airy's tides, &c. are thus locked up."

The first 120 pp. of Gow's *History of Greek Mathematics* give an elaborate survey of the whole subject. Robertson Smith's article "Numeral" in vol. 19, *Encyc. Brit.*, is extremely interesting. We await Sir T. L. Heath's forthcoming volume on Greek Mathematics for the latest presentation of the subject.]

## IV.

I. i. 401. **Apocrypha.**—What editions of the Bible containing the Apocrypha are now on sale in the ordinary way ? M.  
[To this no answer appeared.]

## V.

I. i. 401. **Theses.**—Many German works introduced into catalogues, are *theses* defended at the universities. The name of the *President* is generally first, and in larger letters than that of the proponent, who is usually the author. Hence, it often happens, that the *Thesis* is entered as a work written by the *Praeses*. But it not infrequently happened, that this *Praeses* was *really* the author; and that, as an easy way of publishing his thoughts, he entrusted an essay to a candidate for a degree, to be defended by him. The seventh rule of the Museum Catalogue runs thus :

"The respondent or defender in a thesis to be considered its author, except when it unequivocally appears to be the work of the *Praeses*."

Now, I would ask, what are the usual signs of the authorship ? Are there any catalogues of Theses ? Any bibliographical works which contain hints for guidance in this matter ? Any correspondents who can advise generally on the whole matter ? M.

I. i. 461. Perhaps it may assist your correspondent "M" to be informed that the University of Göttingen is particularly rich in "*Theses*" (termed *Disputationes et Dissertationes*), to which there is a large room entirely devoted in the library of that university; together with the transactions of learned bodies. A special librarian is attached to this department, which is much consulted. A Catalogue was begun to be published of this collection, so far as respects the *Memoirs* contained in the various transactions, in 1801, by J. D. Reuss; and 16 vols. in 4to. had appeared up to 1821; after which, I believe, the publication has been suspended. Of Catalogues of Theses, I think the following work is in good esteem :—*Dissert. Acad. Upsal. habitae sub Praesid. C. P. Thunberg*, 3 tom. 8vo. Götting. 1799-1801. The second part of vol. ii. in the *Catalogus Bibliothecae Thottianae* (7. vol. 8vo. Fauniae, 1789-1795) contains a catalogue, which it might be well to consult, of dissertations on subjects of jurisprudence. I find it to be more frequently the case to place Theses or dissertations under the name of the president or head of the institution or college where

they were delivered, than under the writer's name. At least, in a collective sense the former method is adopted, as in the following instance: Schultens (Alb.), *Sylloge Dissertationem Philologico-Exegeticarum, adversus Auctoribus Editarum, sub Praesidio A. Schultens, etc.*, 2 tom.: although if the author should happen to be distinguished for his other productions, all that he wrote is anxiously sought out, and placed under his own name.

J. M.

The Editor adds: "M" may also be referred to the *Catalogus Dissertationum Academicarum quibus nuper aucta est Bibliotheca Bodleiana*, a quarto volume printed at the Oxford University Press in 1834.

#### VI.

I. i. 401. **J. B.'s Treatise on Art and Nature.**—By a scrap of a book, apparently of the sixteenth century, it seems to be a Treatise by J. B. upon Art and Nature: the first book is "of Water-works." What book is this?

M.

"Hermes" (p. 458) suggests that this volume is "The Mysteries of Nature and Art, in Foure Severall Parts: The First of Water Works—the Second of Fire Works, etc., etc. By John Bate. 2nd edition. 1635. Contains a rude engraving of the author: "Vera effigies Johannis Bate, memoria manet, modo permanent studium et industria."

[Bate] J(ohn). *The Mysteries of Nature and Art: contained in foure severall Treatises, the first of Water Workes, the second of Fyer Workes, and the third of Drawing, Colouring, Painting and Engraving, the fourth of Divers Experiments, as wel serviceable as delightful: partly collected, and partly of the Authors Peculiar Practice, and Inventions.* 1634. Imprinted for Ralph Mab and are to be sold by John Jackson and Francis Church at the Kings armes in Cheapside, 1634, sm. 4to.

Bate is not in the *D.N.B.* S. states that there is no copy in the British Museum, and that this volume contains instructions how to make Weather-glasses; Water-clocks, etc., Experiments of "drawing water by the Crane, and by Engins, of forcing water by ayre compressed and by Engins, of producing sounds by ayre and water, by evaporation of water by fire, and by Engins, of motions by evaporating water, and by rarifying ayre." S. also draws attention to instructions (on p. 155) for making a Freezing Mixture, and points out that Robert Boyle has been credited with making the first experiments with that object in 1667—thirty-three years after this volume was published.]

#### VII.

I. i. 402. **Latin Names of Towns.**—A correspondent who answered the Query as to the "Latin Names of Towns" in titles [of books], referred your readers to the Supplement of Lemprière. I am much obliged to him for the hint, and have obtained the work in consequence; but it is right your readers should know that the information therein given must only be taken as suggestive, and sometimes as dismissible on reference to the commonest gazetteer. I opened at the letter N; and found, that of three entries, the first my eye lighted upon, two were palpably wrong. The first informs us that "Naeostadium in Palatinatu" is in "France," the third that "Nellore is in Ceylon." I am bound to say that I do not find errors so thickly scattered throughout, and that the list will be useful to me. But, Query, is there anything extensive of which the accuracy can be depended upon?

M.

R. G. suggested an anonymous octavo volume, *Dictionnaire Interprète-manuel des Noms Latins de la Géographie ancienne et moderne*. . . . Paris, 1777.

[Naeostadium in Palatinatu is probably Neustadt in the Bavarian palatinate, now a centre of the Rhenish "grape-cure." Nellore is a town and district in the Madras Presidency.]

## VIII.

I. i. 462. **Note Books.**—Looking at what your correspondent says about "Note Books," I think the following hint may be useful to others, as it has been to myself. Many persons never get so far as the formality of a common-place book, and do not like to write in their books. Let them follow my plan. The envelope maker will procure them any number of little slips of white paper, with a touch of isinglass at each of the four corners. Let the note be written on one of these, and then let the slip be stuck into any book which is sure to be wanted in connection with the subject when it comes up again; either by one, two, or four corners, as convenient. The isinglass will not hurt the book, if ever it be wanted to remove the slip. A note is more in the way, when attached to a book which suggested it, than when buried among unindexed miscellanies; and there are few who index themselves. Your motto is good, as far as it goes; but the other half is wanting:

"When made a note of,—find if you can."

M.

[The motto on title-page of "N. & Q." is: "When found, make a note of."]

—CAPTAIN CUTTLE.

The opportunity is tempting. We make two.

*Dombey and Son*, c. xv. "In the Proverbs of Solomon you will find the following words, 'May we never want a friend in need, nor a bottle to give him!' When found, make a note of."

Captain Cuttle had a hook on his right wrist. But in c. xxiii. we read: "Clara a hoy!" cried the Captain, putting a hand to each side of his mouth.]

## IX.

I. i. 492. **Vox Populi Vox Dei.**—Your correspondent "Quaesitor" asks for the origin of the saying *Vox Populi Vox Dei*. Warwick, in his *Spare Minutes* (1637) says: "That the voice of the common people is the voice of God, is the common voice of the people; yet it is as full of falsehood as commonnesse. The cry before Pilate's judgment-seat, 'Let him be crucified,' was *vox populi*, 'the cry of all the people.' How far was it the voice of God?"

M.

The Editor added the opinion of Sir G. Cornewall Lewis, as given in his *Essay on the Influence of Authority in Matters of Opinion*, that this proverb "in its original sense, appears to be an echo of some of the sentences in the classical writers, which attribute a divine or prophetic character to common fame or rumour."

[Mr. W. F. H. King's *Classical and Foreign Quotations*, 1904, states that Geo. Büchmann (1822-1884) in his *Geflügelte Worte*, 19th ed., 1898, pp. 324-5, quotes a passage from Alcuin's *Capitulare admonitionis ad Carolum*, § ix. (Baluzio, *Miscell.*, vol. i. p. 376, Paris, 1678): "Nec audiendi qui solent dicere, *Vox populi, vox Dei*, quum tumultuositas vulgi semper insaniae proxima sit." Thus the saying is earlier than the eighth century. William of Malmesbury (*De gestis pontificum Anglorum*, i. 14) quotes it with reference to Odo's election in 992 A.D. as "illud proverbium." A similar sentiment occurs in Hesiod (*Op.* 761):

φήμη δ' οὐτις πάντα ἀπόλλυται, ἢν τινα πολλοὶ  
λαοὶ φημῖν οὐσι· θεὸς γὰρ τις ἐστὶ καὶ αὐτῇ.

The saying that is voiced by many people never wholly dies, since it is itself divine.]

## X.

I. ii. 21. **Wood Paper.**—The reprint of the *Works of Bishop Wilkins*, London, 1802, 2 vols. 8vo, is said to be on paper made from wood pulp.

It has all the appearance of it in roughness, thickness, and very unequal opacity. Any sheet looked at with a candle behind it is like a firmament scattered with luminous nebulae. I can find mention of straw paper, as patented about the time; but I should think it almost impossible (knowing how light the Indian rice paper is) that the heavy fabric above mentioned should be of straw. Is it from wood? If so, what is the history of the invention, and what other works were printed in it? M.

[For details of manufacture of paper from esparto, straw, and wood pulp, v. *Encyc. Brit.* "Paper," xx. 728-9.]

## XI.

I. ii. 22. **Construe and Translate.**—In my school-days verbal rendering from Latin or Greek into English was *construing*; the same on paper was *translating*. Whence this difference of phrase? M.

C. B. remarked (p. 77) that it is very common, in language, for two words, originally of the same meaning, or two spellings of the same word, to be gradually appropriated to two subordinate uses, applications, and meanings of the word respectively, an accident deciding which of the two is taken for one of the subdivisions, and which for the other. Despatch and dispatch are two such appropriations of recent date. "Construe" originally must have meant "explain the construction" (cf. what is called *syntax*), just as with respect to a single word we use "parse."

[The *N.E.D.* gives the following stages of change of *construe* :

To form by putting together materials; to combine words or parts of speech grammatically; to analyse or trace grammatical construction of a sentence—to take words in order, to show meaning—to do this in foreign and especially classical languages, adding word for word translation; hence, loosely, to translate orally. 1362. Langland.]

## XII.

I. ii. 72. **Morganitic Marriage.**—In Ducange, etc., the adjective *morganitic* is connected with the *morgengab* (morning gift), which was usual from a husband to his wife the day after their marriage. How comes this adjective to be applied to marriages in which the wife does not take her husband's rank? M.

[The form *morganitic* is unusual, and cannot be justified. In the Index to *N. & Q.* I. ii. the word is entered as *morganatic*. The *N.E.D.* gives Med. Lat. *morganaticum*; O.H.G. *morgengabe*. A distinctive epithet for the marriage of a man (or woman) of high rank and a woman (or man) in a lower station. From *morganaticum* we have *morganatisch* (G.) and *morganatique* (F.). The complete phrase was *matrimonium ad morganaticam*, where the last word is probably the "morning-gift." Du Cange is quoted to show that neither the lady nor the children born of such a union shared in anything but the morning gift. The adjective *morganatic* first appears in Chambers's *Encyclopaedia* (1727-1741).]

## XIII.

I. ii. 74. "A Frog he would A-Wooing go." **Rowley Powley.**—As generally inclined to the belief that everything is older than anybody knows of, I am rather started by "Rowley Powley" not being as old as myself. I remember seeing mentioned somewhere, without any reference to this chorus, that *rowley powley* is a name for a plump fowl, of which both "gammon and spinach" are posthumous connexions. I cannot help thinking that this may be a clue to some prior occurrence of the chorus, with or without the song. If "derry down," which has

been said to be Druidical, were judged of by the last song it went with, how old would be the Druids ? M.

[The *N.E.D.* gives Rowley-Powley as a form of Roly-Poly, in its turn a fanciful formation from Roll. It had the following meanings: (1) 1601, rascal; 1605, Pall-Mall; 1679, trifling, worthless; (2) a game in which the ball rolls on to a certain spot—roulet; (3) children on the hillside; (4) a pea, 1784, Here's your large rowley-powleys 6d. a packet; (5) Pudding, 1848; (6) Roll or coil of hair, 1866; (7) a kind of dance, 1830, Carleton's *Traits of the Irish Peasantry*; (8) Short, stout, podgy, 1820 (Miss Mitford); 1853 (Dinah Muloch). "Old Rowley" was a nickname of Charles II.]

## XIV.

I. ii. 136. **The Frog and the Crow of Ennow**—I should be glad to get an answer to the following query from some one of your readers: I remember some few odd lines of a song I used to hear sung many years ago, and wish to learn anything as regards its date, authorship,—indeed, any particulars; and where I shall be likely to find it at length. What I remember is:

"There was a little frog in the river Swim-o,  
And there was an old crow lived in the wood of Ennow,  
Come on shore, come on shore, said the frog to the frog again-o;  
Thank you, sir, thank you, sir, said the frog to the crow of Ennow.

But there is sweet music under yonder green willow,  
And there are the dancers, the dancers in yellow."

M.

T. I. (p. 222) gave the form of "The Frog and the Crow" with which he had been familiar since childhood. He stated that it was tolerably well known in Lancashire, and that the *point* consists in giving a scream over the last "oh!" which invariably, if well done, elicits a start even in those who know what to expect.

*The Frog and the Crow.*

"There was a jolly fat frog lived in the river Swimmo,  
And there was a comely black crow lived on the river Brimmo;  
Come on shore, come on shore, said the crow to the frog, and then, oh;  
No, you'll bite me, no, you'll bite me, said the frog to the crow again, oh.

"But there is sweet music on yonder green hill, oh,  
And you shall be a dancer, a dancer in yellow;  
All in yellow, all in yellow, said the crow to the frog, and then, oh;  
Sir, I thank you, sir, I thank you, said the frog to the crow again, oh.

"Farewell, ye little fishes, that are in the river Swimmo,  
For I am going to be a dancer, a dancer in yellow;  
Oh, beware, Oh, beware, said the fish to the frog again, oh;  
All in yellow, all in yellow, said the frog to the fish, and then, oh.

"The frog he came a-swimming, a-swimming, to land, oh,  
And the crow, he came a-hopping to lend him his hand, oh;  
Sir, I thank you; Sir, I thank you, said the frog to the crow, and then, oh;  
Sir, you're welcome; Sir, you're welcome, said the crow to the frog again, oh.

"But where is the music on yonder hill, oh;  
But where are the dancers, the dancers in yellow,  
All in yellow, all in yellow? said the frog to the crow, and then, oh;  
Sir, they're here; Sir, they're here, said the crow to the frog, and eat him all up, OH" (screamed).

(To be continued.)

MATHEMATICAL NOTES.

512. [D. 2. c ; 6. c. γ.] The following is a method of proof for the infinite product expressions for  $\sin \theta$  and  $\cos \theta$ , and for the series related to them, which may be of interest. It is a development of a well-known proof of Wallis's formula.

Let

$$u_m = \int_0^{\frac{1}{2}\pi} \cos k\theta \cos^m \theta d\theta,$$

$$a_m = \int_0^{\frac{1}{2}\pi} \cos^m \theta d\theta.$$

Then, if  $k$  is less than unity and real,

$$1 > \cos k\theta > \cos \theta$$

for all values of  $\theta$  between 0 and  $\frac{1}{2}\pi$  inclusive, and thus

$$a_m > u_m > a_{m+1}.$$

Now, if  $m$  is a positive integer,  $u_m$  can be found by a formula of reduction as follows :

$$\begin{aligned} \frac{d}{d\theta} \{m \cos k\theta \cos^{m-1} \theta \sin \theta - k \sin k\theta \cos^m \theta\} \\ = \cos k\theta \{m \cos^m \theta - m(m-1) \cos^{m-2} \theta \sin^2 \theta - k^2 \cos^m \theta\} \\ = \cos k\theta \{(m^2 - k^2) \cos^m \theta - m(m-1) \cos^{m-2} \theta\}. \end{aligned}$$

By integrating from 0 to  $\frac{1}{2}\pi$ , it follows that when  $m > 1$ ,

$$u_m = \frac{m(m-1)}{m^2 - k^2} u_{m-2},$$

and that

$$\begin{aligned} (1^2 - k^2) u_1 &= \cos \frac{1}{2} k\pi, \\ -k^2 u_0 &= -k \sin \frac{1}{2} k\pi. \end{aligned}$$

Hence

$$\begin{aligned} u_{2m} &= \frac{2m!}{k(2^2 - k^2)(4^2 - k^2) \dots (4m^2 - k^2)} \sin \frac{1}{2} k\pi, \\ u_{2m+1} &= \frac{(2m+1)!}{(1^2 - k^2)(3^2 - k^2) \dots \{(2m+1)^2 - k^2\}} \cos \frac{1}{2} k\pi. \end{aligned}$$

Now, since

$$a_{2m-1} > u_{2m-1} > a_{2m} > u_{2m} > a_{2m+1}$$

and

$$a_{2m+1}/a_{2m-1} = 2m/(2m+1),$$

it follows that  $u_{2m-1}/a_{2m-1}$  and  $u_{2m}/a_{2m}$  can both be made as near to 1 as we please by increasing  $m$ .

But

$$\frac{u_{2m-1}}{a_{2m-1}} = \frac{1^2 \cdot 3^2 \dots (2m-1)^2}{(1^2 - k^2)(3^2 - k^2) \dots \{(2m-1)^2 - k^2\}} \cos \frac{k\pi}{2}$$

and

$$\frac{u_{2m}}{a_{2m}} = \frac{2^2 \cdot 4^2 \dots (2m)^2}{k(2^2 - k^2) \dots (4m^2 - k^2)} \frac{2}{\pi} \sin \frac{1}{2} k\pi,$$

so that we have, by putting  $\theta$  for  $\frac{1}{2}k\pi$ ,

$$\begin{aligned} \sin \theta &= \theta \left(1 - \frac{\theta^2}{\pi^2}\right) \left(1 - \frac{\theta^2}{2^2 \pi^2}\right) \dots \text{to } \infty, \\ \cos \theta &= \left(1 - \frac{2^2 \theta^2}{\pi^2}\right) \left(1 - \frac{2^2 \theta^2}{3^2 \pi^2}\right) \dots \text{to } \infty. \end{aligned}$$

Wallis's formula is proved by the fact that  $a_{2m}/a_{2m-1}$  tends to the limit when  $m$  is increased indefinitely.

It has been assumed that  $k$  is real and less than unity. If this is not so, the above proof may be modified as follows:

$$\begin{aligned} a_m - u_m &= \int_0^{\frac{1}{2}\pi} (1 - \cos k\theta) \cos^m \theta d\theta \\ &= \int_0^{\frac{1}{2}\pi} 2 \sin^2 \frac{1}{2} k\theta \cos^m \theta d\theta. \end{aligned}$$

Thus  $|a_m - u_m| \leq \int_0^{\frac{1}{2}\pi} g \sin^2 \theta \cos^{m-2} \theta d\theta,$

where  $g$  is a finite constant quantity, namely, the greatest value of  $|2 \sin^2 \frac{1}{2} k\theta| \div \tan^2 \theta$  when  $0 \leq \theta \leq \frac{1}{2}\pi$ .

We have then  $|a_m - u_m| \leq g(a_{m-2} - a_m),$   
which  $= g a_m / (m-1).$

Thus  $u_m/a_m$  again tends to 1 when  $m$  is increased indefinitely, and therefore the infinite products for  $\sin \theta$  and  $\cos \theta$  hold universally.

Again, if  $s_1 = \frac{1}{k} + \frac{2k}{k^2 - 2^2} + \frac{2k}{k^2 - 4^2} + \dots + \frac{2k}{k^2 - 4m^2}$

and  $s_2 = \frac{1}{k^2} + \frac{1}{(k-2)^2} + \frac{1}{(k+2)^2} + \dots + \frac{1}{(k-2m)^2} + \frac{1}{(k+2m)^2},$

we have  $\frac{1}{2}\pi \cot \frac{1}{2}k\pi - s_1 = \frac{1}{u_{2m}} \left( \frac{d}{dk} u_{2m} \right),$

$$\frac{1}{4}\pi^2 \operatorname{cosec}^2 \frac{1}{2}k\pi - s_2 = -\frac{1}{u_{2m}} \frac{d^2}{dk^2} u_{2m} + \left( \frac{1}{u_{2m}} \frac{d}{dk} u_{2m} \right)^2.$$

Now \*

$$\frac{d}{dk} u_{2m} = - \int_0^{\frac{1}{2}\pi} \theta \sin k\theta \cos^{2m} \theta d\theta,$$

$$\frac{d^2}{dk^2} u_{2m} = - \int_0^{\frac{1}{2}\pi} \theta^2 \cos k\theta \cos^{2m} \theta d\theta,$$

and each of these, since  $\theta < \tan \theta$ , is numerically less than

$$h \int_0^{\frac{1}{2}\pi} \sin^2 \theta \cos^{2m-2} \theta d\theta, \text{ that is, } h a_{2m} / (2m-1),$$

where  $h$  is a finite quantity independent of  $m$ , namely, the greatest absolute value of  $\sin k\theta / \tan \theta$  or of  $\cos k\theta$ .

(When  $k$  is real and between 0 and 1,  $h$  may be taken as 1.)

Thus  $\frac{1}{u_{2m}} \frac{d}{dk} u_{2m}$  and  $\frac{1}{u_{2m}} \frac{d^2}{dk^2} u_{2m}$  both tend to zero as  $m$  increases, that is,  $s_1, s_2$  tend to the limits  $\frac{1}{2}\pi \cot \frac{1}{2}k\pi, \frac{1}{4}\pi^2 \operatorname{cosec}^2 \frac{1}{2}k\pi$ , and

$$\cot \theta = \frac{1}{\theta} + \frac{2\theta}{\theta^2 - \pi^2} + \frac{2\theta}{\theta^2 - 4\pi^2} + \dots \text{ to } \infty,$$

$$\begin{aligned} \operatorname{cosec}^2 \theta &= \frac{1}{\theta^2} + \frac{1}{(\theta + \pi)^2} + \frac{1}{(\theta + 2\pi)^2} + \dots \text{ to } \infty \\ &\quad + \frac{1}{(\theta - \pi)^2} + \frac{1}{(\theta - 2\pi)^2} + \dots \text{ to } \infty. \end{aligned}$$

The formulae

$$\operatorname{cosec} \theta = \cot \frac{1}{2}\theta - \cot \theta, \quad \tan \theta = \cot \theta - 2 \cot 2\theta, \quad \sec \theta = \operatorname{cosec} \left( \theta + \frac{\pi}{2} \right)$$

may now be used to find the series for  $\operatorname{cosec} \theta, \tan \theta, \sec \theta$  as usual.

A. C. DIXON.

\* The differentiations under the integral sign could be avoided by means of separate formulae of reduction for

$$\int \theta \sin k\theta \cos^m \theta d\theta \quad \text{and} \quad \int \theta^2 \sin k\theta \cos^m \theta d\theta.$$

513. [L. 1. c.] *Note on Pascal's Theorem.*

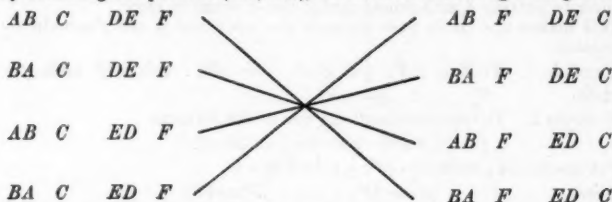
The joins of opposite sides of a hexagon inscribed in a conic are collinear. Thus, if  $ABCDEF$  be a hexagon and if  $AB, DE$  meet in  $X$ ,  $BC, EF$  in  $Y$  and  $CD, FA$  in  $Z$ ;  $XYZ$  is a straight line, called a Pascal Line.

There are 60 Pascal lines corresponding to the different hexagons, which may be obtained by taking the 6 points on the conic in all possible orders. This may easily be seen by considering the well-known problem of the number of arrangements of 6 different beads on a string so as to form a necklace.

These 60 lines intersect in fours in 45 points.

First consider the hexagon  $ABCDEF$ .

By retaining the point  $X$  ( $AB, DE$ ), we have the following arrangements:



The second set obviously gives the same four hexagons as the first, thus the hexagons  $ABCDEF$  and  $BAFEDC$  are the same.

Hence 4 different Pascal lines pass through the point  $X$ .

There are 3 points on each line and 60 lines.

Hence there are  $3 \times 60 \div 4 = 45$  points.

The same result may be obtained as follows:

To obtain the number of points we notice that a point may be represented by the notation  $(AB, CD)$ .

Now the number of combinations of 6 things 4 at a time = 15.

Let  $ABCD$  be one of these. These 4 letters give 3 points, since  $A$  may be paired with either  $B, C$  or  $D$ , leaving the remaining two for the other pair.

Then  $(AC, BD)$  represents a point.

Hence the number of points =  $15 \times 3 = 45$ .

Now let  $x$  be the number of lines which pass through one point; we have, equating the number of lines,

$$45 \times x = 60 \times 3,$$

since each Pascal line may be taken as 3 coincident lines forming the triangle  $XYZ$ .

Hence

$$x = 4.$$

We have of course the corresponding arrangement of Brianchon points, viz. that the 60 Brianchon points given by 6 tangents to a conic lie in fours on 45 lines. The proof of this may be obtained by interchanging point for line and line for point in the above.

T. WORRELL.

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514. [X. 4. b. a.] *Graphical Solution of a Biquadratic.*

I. The note under the above heading on p. 337, vol. v., by an anonymous contributor is to this effect:

The equation  $x^4 + 4ax^3 + \beta x^2 + \gamma x + \delta = 0$

is satisfied by the abscissae of the intersections of the parabola

$$ky = (x + a)^2$$

and the circle

$$x^2 + y^2 - 2gx - 2fy + c = 0,$$



515. [K. 2. d.] (Vol. vi. p. 153, No. 349.) To prove  $4R \cdot QN = SA \cdot S'A$ .

Let  $\alpha, \beta, \gamma$  be points of contact of in-conic with the sides of  $\triangle ABC$ .

Draw  $P\delta R$  the tangent parallel to  $BC$ .

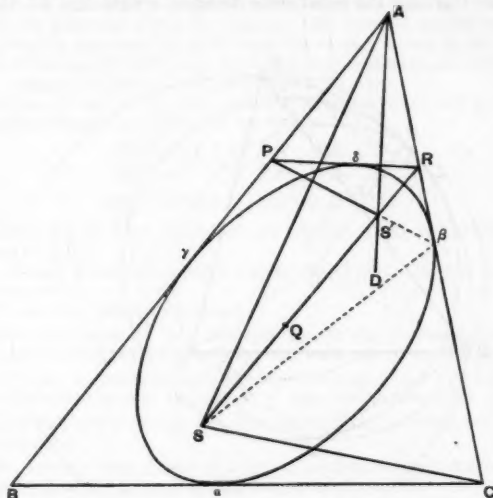
Then we have, by elementary Geometrical Conics,

$$\angle PS'R = \frac{1}{2}(\angle \gamma S'\delta + \angle \delta S'\beta) = \frac{1}{2}(360^\circ - \angle \gamma S'\beta) = 180^\circ - \angle DS'\beta = \angle AS'\beta;$$

$$\therefore \angle AS'P = \angle RS'\beta = \angle SC\beta, \text{ since } \triangle RS'\beta, SC\beta \text{ are similar,}$$

$$\text{also } \angle PAS' = \angle SAC;$$

$$\therefore \triangle PAS', SAC \text{ are equiangular and } SA \cdot S'A = AC \cdot AP. \dots\dots(1)$$



Now if  $AA'$  (the perpendicular from  $A$  on  $BC$ ) meet  $PR$  in  $K$ ,

$$AK = 2QN \text{ and } \frac{R_1}{R} = \frac{AK}{AA'} = \frac{AR}{AC},$$

where  $R_1$  is circumradius of  $\triangle APR$ ; then

$$4R \cdot QN = 2R \cdot AK = \frac{2R}{2R_1} \cdot AP \cdot AR, \text{ (since } AP \cdot AR = AK \cdot 2R_1) = AC \cdot AP; (2)$$

$$\therefore SA \cdot S'A = 4R \cdot QN.$$

E. P. LEWIS.

516. [K. 2. d.] Pedal Circles.

*Def.* The pedal circle of a point with respect to a triangle is the circle which passes through the feet of the perpendiculars drawn from the point to the sides of the triangle.

*Def.* The point is termed a pole of its pedal circle.

*Theo.* Every pedal circle has two poles, and these poles are isogonal conjugates with regard to the triangle.

Let  $PL, PM, PN$  be respectively perpendicular to  $BC, CA, AB$ . Let the pedal circle  $LMN$  cut these sides respectively again in  $L'M'N'$ . It is easily proved that the perpendiculars at  $L', M', N'$  respectively to  $BC, CA, AB$  meet in a point  $P'$ , which is such that  $O$ , the centre of the pedal circle, is the middle point of  $PP'$ .

Again, if  $\rho$  denote the radius,

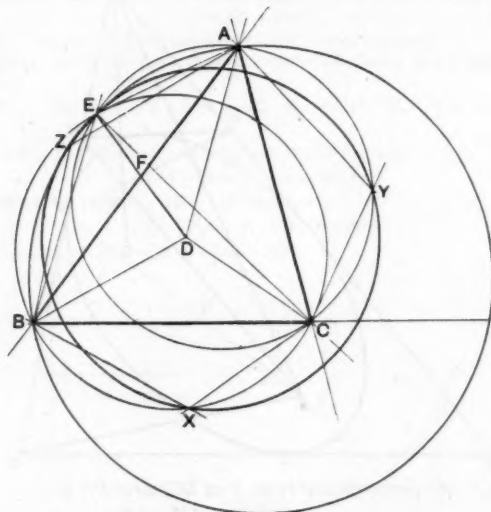
$$PL \cdot PL' = PM \cdot PM' = PN \cdot PN' = \rho^2 - PO^2.$$

$\therefore P$  and  $P'$  are isogonal conjugates with regard to  $ABC$ .

*Examples.* (1) The nine-point circle is a pedal circle whose poles are the circumcentre and the orthocentre.

(2) Any circle which touches the sides of a triangle is a pedal circle with two coincident poles.

(3) When  $P$  is on the circumcircle of  $ABC$ , its isogonal conjugate is at infinity. In this case the pedal circle becomes a right line, i.e. the Simson line.



*Theorem.* The four pedal circles of each of four points with respect to the triangle formed by the other three are concurrent.

Let the four points be  $A, B, C, D$ .

Let  $X, Y, Z$  be the reflections of  $D$  with regard respectively to  $BC, CA, AB$ .

It is a well-known theorem, easily established by Euc. III. 21, 22 and converses, that the four circles  $BXC, CYA, AZB$ , and  $XYZ$  meet in a point.

Let us call this point  $E$ .

Now, regarding  $D$  as a centre to which the other points approach half-way, we see that

The nine-point circles of  $BDC, CDA, ADB$ , and the pedal circle of  $D$  with regard to  $ABC$  all meet in the middle point of  $DE$ .

From this we infer easily that

The four nine-point circles of  $BDC, CDA, ADB, ABC$  meet in the middle point of  $DE$ .

The four pedal circles  $A-BCD, B-CDA, C-DAB, D-ABC$ , meet in the middle point of  $DE$ .

When  $A, B, C, D$  form a concyclic system, the above theorem reduces to the well-known one of the four Simson lines.

Let  $F$  be the middle point of  $DE$ .

$F$  is plainly the centre of the rectangular hyperbola which passes through  $ABCD$ .

And, from the geometry of conics, there also passes through  $F$  another circle, i.e. that passing through the intersections  $BC - AD, CA - BD, AB - CD$ .

Thus there are concurrent nine circles related to the figure formed by four points.

On this account,  $F$  might be termed the "Nine-Circle Point" of the quadrilateral  $ABCD$ .

We shall now consider the points  $D$  and  $E$  as related to the triangle  $ABC$ .

$D$  and  $E$  are points at which the sides of  $ABC$  subtend similar angles.

Hence triangles reciprocal to  $ABC$  from the origins  $D$  and  $E$  are similar.

Now the reciprocal of  $ABC$  from the origin  $D$  is similar to the pedal triangle of  $D$ , the isogonal conjugate of  $D$  with respect to  $ABC$ .

If, therefore,  $D'$  and  $E'$  be the isogonal conjugates of  $D$  and  $E$ , it follows that the pedal triangles of  $D'$  and  $E'$  are similar.

$$\text{Hence} \quad \frac{BC \cdot AD'}{BC \cdot AE'} = \frac{CA \cdot BD'}{CA \cdot BE'} = \frac{AB \cdot CD'}{AB \cdot CE'}$$

or

$$AD' : AE' = BD' : BE' = CD' : CE'.$$

From which it follows that  $D'$  and  $E'$  are inverse points with regard to the circumcircle of  $ABC$ .

That is,  $D$  and  $E$  are the isogonal conjugates of points inverse with regard to the circumcircle.

We have now the general theorem :

The nine-point circle of any triangle bisects the distance between the isogonal conjugates of points inverse with regard to the circumcircle.

When we take as inverse points the circumcentre and a point at infinity, the above general theorem reduces to a very well-known one, i.e.

The nine-point circle bisects any line drawn from the orthocentre to the circumcircle.

Let us for brevity term  $D$  and  $E$  Angular Conjugates with regard to  $ABC$ , since the sides of  $ABC$  subtend similar angles at  $D$  and  $E$ .

We have proved that

The nine-point circle of  $ABC$  intersects the pedal circle of  $D$  in the middle point of the line joining  $D$  and its angular conjugate.

Now  $D$  and  $D'$  have the same pedal circle. Hence the nine-point circle also intersects the pedal circle of  $D$  in the middle point of the line joining  $D'$  and its angular conjugate.

We have pointed out that any circle which touches the three sides of a triangle is a pedal circle with two coincident poles.

In this case  $D$  and  $D'$  coincide. Therefore their angular conjugates also coincide. And therefore, from above, the two points of intersection of the pedal and nine-points circles coincide.

Hence Feuerbach's Theorem :

Any circle which touches the three sides of a triangle touches also the nine-point circle.

From this we see that, if  $D$  is the centre of any circle touching the three sides of  $ABC$ ,  $F$  the nine-circle point of  $ABCD$  is the point of contact of that circle with the nine-point circle of  $ABC$ .

Hence the theorem :

The circle, which passes through the intersections with the opposite sides of three concurrent bisectors of the angles of a triangle, passes also through the point of contact of the nine-point circle with the corresponding circle touching three sides.

The following theorems can now be established easily:

(1)  $A, B, C, D$  are four points.

$A'$  is the angular conjugate of  $A$  with respect to  $BCD$ ,  $B'$  of  $B$  with respect to  $CDA$ , etc.

The figure  $A'B'C'D'$  is identically equal to the figure  $ABCD$ .

(2)  $I$  is the incentre of  $ABC$ , and  $Q$  is the inverse of  $I$  with respect to the circumcircle of  $ABC$ .

$Q'$  is the isogonal conjugate of  $Q$ .

Prove that  $IQ' = 2r$ ; and that  $I, Q'$ , and the centre of the nine-point circle are collinear.

(3) If two points are inverse with regard to the circumcircle, their pedal circles intersect on the nine-point circle.

(4) If  $D$  and  $E$  are angular conjugates with respect to  $ABC$ , then the nine-circle points of  $ABCD$  and  $ABCE$  coincide.

J. H. LAWLER.

4 "In one of Charnac's Mathematical Tables a line fell out before going to press, and instead of being replaced at the bottom of the page it was put in at the top, thus causing twenty-six errors. Besides these, however, only ten errors have been found in the whole work of 1020 pages, all full of figures. Vieta's *Canon Mathematicus* (1579) is of great rarity, from the author being discontented with the misprints that had escaped his notice,\* and on that account withdrawing or re-purchasing all the copies he could meet with. Some mathematicians, to ensure accuracy, have made their calculations with the types in their own hands. [This must be the origin of the phrase to have one's hands full.]"—*Literary Blunders*, by H. B. Wheatley, 1893, p. 144.

"In connection with figures may be mentioned the terrible confusion which is caused by the simple dropping out of a decimal point. Thus a passage in which 6'36 is referred to naturally becomes utter nonsense when 636 is printed instead. Such a misprint is as bad as the blunder of the French composer, who, having to set up a passage referring to Captain Cook, turned 'de Cook' into 'de 600 kilos.'"—*Loc. cit.* p. 145.

[CHARNAC, Ladislaus, *Cribrum Arithmeticum; sive tabula continens numeros primos* ... Davenport, 1811, 4to, pp. 1020. Of this Dr. Glaisher writes in his invaluable "Report on Mathematical Tables" (*Report of the British Association*, 1873, p. 37): "In a thick quarto are given all the simple divisors of numbers from 1 to 1,020,000 (multiples of 2, 3, and 5 being excluded). This book was found by Burekhardt (who subsequently published the same table, the least divisor only being given) to be very accurate: he detected only 38 errors (he has given them in the preface to his first million), of which only 9 are due to the author, the remaining 29 having been caused by the slipping, etc., of type, in the printing."]

5. Who was it who wrote of Napoleon: "Esprit de feu pour l'Algèbre, et de glace pour le Latin"? Or as some one translated it:

"His fiery soul for mathematics burned,  
But turned to ice when Virgil's page he turned."

6. Epitaph in Debtling Churchyard:

Alas!  
Poor Yorke.  
1837.

Yorke was an eccentric individual, who lived a hermit's life in Debtling, believed in the theory of perpetual motion, and made many fruitless attempts to carry it out.

\* "Infelicitè editus," says Vieta (Schorten's edition, p. 323, quoted by De Morgan, *Vieta*, Penny Cyclopædia).

## REVIEWS.

**Edinburgh Mathematical Tracts.** General Editor, PROF. E. T. WHITTAKER, F.R.S. (G. Bell & Sons.)

1. **Descriptive Geometry and Photogrammetry.** By E. LINDSAY INCE. 2s. 6d. net.

2. **Interpolation and Numerical Integration.** By DAVID GIBB. 3s. 6d. net.

6. **Solution of Spherical Triangles.** By HERBERT BELL. 2s. 6d. net.

These tracts would seem to serve a very useful purpose.

In No. 1 we find a very careful exposition of what is generally known as "Practical Solid Geometry," a chapter on Perspective and one on Photogrammetry, or the art of reading plans and elevations from two photographs of the same district from different points of view. The whole is well-written and illustrated with clear diagrams. The only remark that we feel called upon to make is that of surprise that there is not more made of the connection between the matter and that of "figured plans."

No. 2 is an altogether more abstruse compilation. It opens with an exposition of the chief theorems in the Calculus of Finite Differences; the second chapter is on Interpolation, and gives all the information that can be required with regard to the well-known (?) formulae, the explanation being as lucid as the nature of the subject will admit; the third chapter is upon the construction and use of mathematical tables, in which a clear account of the best-known methods is given—the author does not appear to know the approximate formula for  $\log(1+x)$ , in the form of the fraction  $(6x + 3x^2)/(6 + 6x + x^2)$ , or of the continued fraction of which it is the third convergent with an error in defect less than  $x^2/180$  (see *Math. Gaz.* viii. p. 294); in conclusion we have a chapter on numerical integration, to which the other chapters have been necessary preliminaries. On page 85, we think that the author is hardly doing justice to Simpson's rules by taking such wide intervals as would never be used by a student of practical mathematics; if the intervals are taken merely half the width, the result is 1.947 for the parabolic rule, which is only 0.05 per cent. in error.

There would not seem to be so much demand for the contents of No. 6, although the reviewer is personally glad to see its publication. Teachers of Engineering seem to prefer to give their students the rule-of-thumb principles of Vector Analysis (in 20 years I have never had a student who understood anything at all about it, although I have had one or two who could use it), rather than to teach them the fundamental principles of the trigonometry of dihedral angles. All that the engineer wants as a general rule can be given in a lesson of two hours, and it can be given in a form that he understands. From a theoretical point of view, the training in manipulation of trigonometrical functions is enough to recommend it to any teacher. This volume closes with some account of the subject of nomography; it would have been better if it had been extended to give, say, an example and an explanation of the method as applied to the evaluation of such algebraical formulae as  $V = kd^2h$ . J. M. C.

**The Geometrical Lectures of Isaac Barrow.** By J. M. CHILD. 4s. 6d. net. 1916. (Open Court Publishing Company.)

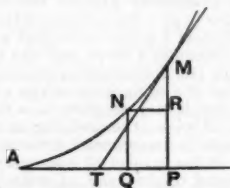
This is an interesting essay in the history of the Calculus, and is worthy of careful perusal by the student of the history of mathematics. The Geometrical Lectures do not seem to have been actually delivered as Lucasian lectures at Cambridge, but they were published along with the second edition of the *Lectiones Opticae* in 1670. The edition before us consists of a translation of the Lectures abridged to such an extent as would admit of the compression of the whole book (including preface, introduction, and running commentary) into a convenient volume of 200 pages. The editor tells us that, as the result of a careful study of the whole book, and especially after turning some of the purely geometrical work into algebraical form, he came to the conclusion

that Barrow had "got the calculus," and in fact intended the Lectures as a complete treatise on the calculus, notwithstanding that it is, in form, almost entirely geometrical. The following is his thesis: *Isaac Barrow was the first inventor of the Infinitesimal Calculus; Newton got the main idea of it from Barrow by personal communication; and Leibniz also was in some measure indebted to Barrow's work, obtaining confirmation of his own original ideas, and suggestions for their further development, from the copy of Barrow's book that he purchased in 1673.*

The first sentence no doubt requires modification because, apart from the fact that many integrations had been made before Barrow's time, it seems certain that Fermat was the first person to perform an actual differentiation, that is to say, to show how to find the limiting value of  $\{f(x+h) - f(x)\}/h$  when  $h$  is made indefinitely small. This Fermat did by dividing out by  $h$  and then making  $h=0$ . Fermat used the result to find the maximum or minimum values of  $f(x)$ , by equating  $f'(x)$  so obtained to 0. This method of finding maximum and minimum values of  $f(x)$  he then applied to finding tangents to curves, which in practice meant finding the subtangent, corresponding in our notation to  $y/(\frac{dy}{dx})$ . But it seems possible to go so far as to say that

Barrow wrote the first systematic treatise on the differential calculus, while preferring to give it a geometrical form. The result of using the geometrical form is to substitute for the algebraical process of differentiation the operation of finding geometrically the tangent to a curve at any point.

This is done quite generally for any curve in Lecture X. § 11, in the figure of which proposition is found the small triangle which we now call the "differential triangle," and in which the small element of the curve (our  $ds$ ) is one side, while the other two sides are the small increments of the coordinates  $x, y$  between the two extremities of  $ds$ . Barrow only incidentally mentions, and gives a few illustrations of, the alternative method of calculation "frequently used by us"; this he does "on the advice of a friend," and here the rules for algebraic differentiation are pretty clearly put, and in a form which Newton had only to modify very slightly for use in analytical work. Barrow denotes by  $a$  and  $e$  respectively the small increments which we call  $dy$  and  $dx$  (Fermat had used only one of these letters,  $e$ , for the increment of  $x$ ); his other standard abbreviations are  $m$  for our  $y$  and  $t$  for the subtangent. He then explains his method thus: "I compare  $MR, NR$  [our  $dy, dx$ ], and



through them  $MP, PT$ , with one another by means of an equation obtained by calculation; meantime observing the following rules:

"1. While calculating, I neglect all terms in which there is a power of  $a$  or  $e$ , or in which there are products of these (for these terms will have no value).

"2. After the equation has been formed, I neglect all terms consisting of letters denoting known or determinate quantities, or terms in which there is no  $a$  or  $e$  (for those terms, brought over to one side of the equation, will always be equal to zero).

"3. For  $a$  I substitute  $m$  (or  $MP$ ) and for  $e$  I substitute  $t$  (or  $PT$ ). Hence finally the length of  $PT$  will be determined. And if an indefinitely small part of any curve enters the calculation, let a small part of the tangent, properly chosen, or any straight line equivalent to it, be put in the place of the small part of the curve (on account of its indefinite smallness)."

The systematic character of the work is shown by the regular way in which propositions for determining tangents by the use of Cartesian coordinates are followed by similar propositions using polar coordinates for the same purpose. All the important standard differentials are found in their equivalent geometrical form, e.g. the differentials of a power of  $x$ , integral or fractional, and of the trigonometrical ratio  $\tan x$ ; we have also the laws for differentiating the sum, product, and quotient of two functions of  $x$ . Barrow seems also to have been the first to appreciate fully that differentiation and integration are the inverse of each other. Among the curves to which Barrow draws tangents are the cycloid, the conchoid, the *quadratrix*, the cissoid (as particular cases of more general curves), the folium of Descartes, the paraboliforms ( $y^m = a^{m-n}x^n$ ), which figure largely in the book, and hyperboliforms.

Coming now to the propositions containing the equivalent of integrations, we should remember that much more had been done by Barrow's time in the integral than in the differential calculus. The equivalent of most of the common integrals had been given, some by Archimedes, others (to mention only the greater names) by Cavalieri, Fermat, Pascal, Roberval, Wallis. We naturally find, therefore, in Barrow, such integrals as those of any integral power of the variable, of the different trigonometrical ratios and their equivalents. Especially noteworthy is the occurrence of  $\int_a^b \frac{1}{x} dx$ , regarding which

Barrow (like Fermat) implies rather than explicitly states that its value is  $\log a - \log b$ . But naturally, seeing that the treatment is geometrical, the effect of many propositions is to express a required integral in terms of another integral known or unknown, much in the way that Archimedes did in the *Method* (though without the use of the mechanical considerations employed by Archimedes); several of the results are actually equivalent to our "integration by parts."

There are some general theorems on rectification, the method being that of reducing the particular problem to a quadrature. Barrow rectifies the cycloid as a particular case of a more general problem; he also rectifies a parabola, while the cases of the logarithmic spiral and the three-cusped hypocycloid can be readily deduced from his results.

The book suffers from some defects and drawbacks. Most persons who wish to make a thorough study of Barrow will regret that so many of Barrow's proofs and diagrams are omitted, and especially the diagrams, which are often not easy to draw with certainty from the description given. This is, however, not the editor's fault, but is due to the necessity of limiting the size of the book. A more serious defect is the number of mistranslations of the original Latin. The editor modestly says that he is no classical scholar; but it would have been well if he had asked some classical friend to revise the translation of the portions of the book, e.g. the preface and the general explanations, which, not being purely mathematical, are written in idiomatic Latin. The following are a few of the more glaring cases: In the Preface to the Geometrical Lectures *alias, opinor, . . . nihil cogitaturus* is translated "Otherwise, I imagine, I shall be thought little of for . . ." (it should be "Otherwise, I believe, I should never have had any idea of. . ."); *comparere* ("to appear") is taken as meaning "to be prepared"; *libentius* ("more willingly") as "more freely"; *nonnulla sperans in illis haberi quae nec eruditiores piquerit inspicere* is translated "hoping that there is nothing in them that it will displease the more erudite to see" (instead of "hoping that they contain some things, at all events, which even the more learned will not be sorry to look into"). Mr. Child has also (pp. 25-26) translated the Preface to the Optical Lectures, and there are some bad cases here too: *nec tamen ut juris id tui fieret defuerunt auctores* is rendered "nor, that you might yet demand it as your due, were other authorities absent" (instead of "nor were there wanting advisers who urged me to put it in your hands"); *ne quidem horum magnam partem relegere sustinui*, translated "in order that I should not indeed put off the rewriting of the greater part of these things," should be "I could not even endure to read over again a great part of the work"; *verum, quod tenellae matres faciliant, a me depulsum partum amicorum haud recusantium nutriciae curae commisi*,

*prout ipse visum esset, educandum aut exponendum* is rendered "as delicate mothers are wont, I committed to the foster-care of friends, not unwillingly, my discarded child, to be led out and set forth as it might seem good to them," instead of "but, as is not uncommon with delicate mothers, I committed to the foster-care of friends, who willingly accepted the charge, my discarded offspring, to be reared or exposed (i.e. left to its fate) as they might think fit" (*educandum* is evidently confused by the editor with *educendum*, and the point of *exponendum* is missed).

Nor is the editor as strong as might be wished on the historical side. One example will suffice. In two places (pp. 139 and 179) he says that Barrow anticipated Wren in rectifying the cycloid. This statement he supports by quoting Williamson as saying (in the article on "Infinitesimal Calculus" in the *Encyclopaedia Britannica*, *Times* edition) that Sir Christopher Wren rectified the cycloid in 1673. Yet he knows that other authorities, e.g. Rouae Ball, give a different date (1658), and he even says that "Ball is probably the more correct." If, however, the true date was 1658, Barrow did not anticipate Wren. In these circumstances it was imperative to find out the truth as to the date, which would not have been difficult. The story is interesting. In June, 1658, Pascal had sent out a circular letter offering a prize for a solution of certain problems relating to the cycloid. In his *Histoire de la Roulette* Pascal speaks (under date 10 Oct., 1658) of the replies he had received, some of which claimed to have solved the problems, while others did not, but merely contained their authors' first reflections or observations on the cycloid. Among the latter, Pascal says, there was nothing finer than Wren's communication, which not only described an elegant method of finding the area of the roulette (cycloid), but also gave the rectification of the curve and any portion of it. "His proposition" (says Pascal) "is that the length of the curve is quadruple of its axis, of which proposition he only sent the enunciation without the proof; and, as he is the first person who has brought out this result, he is entitled to the honour of the discovery." (Roberval indeed claimed in his *De trochoide* that he had found the length of the arc of a cycloid at the time of his first researches regarding that curve, i.e. 1634-38, but had kept his discovery secret up to the time when Wren also discovered it; Pascal seems to accept this, on the ground that, as soon as the proposition was mentioned to Roberval, he at once produced the complete proof of it, but he admits that in any case this could not rob Wren of the credit.) Wallis in his tract *On the cycloid and bodies generated from it* (published in 1659) says that Wren had shown the proof of his proposition to certain friends about the beginning of July, 1658 (*sub initium Julii anni 1658*), and, as he had assumed the proposition in § 1 of his work, Wallis gave Wren's proof in full as an appendix (see Wallis, *Opera*, I. pp. 533 sq.). Wallis returned to the subject in a communication to the Royal Society dated 4 Oct., 1673 (published in the *Phil. Trans.*, 17 Nov., 1673), where he quotes a letter (undated) from Wren himself beginning as follows:

"Sir,—That I did, in the year 1658, find a Streight line equal to that of a Cycloid, and the parts thereof, was then very well known, not in England only, but in France and Holland. And I have not yet heard of any who do pretend to have known it before I discover'd it; which was the same year acknowledged in Print by those of France. . . ."

It should be added that the notes on and elucidations of the mathematical content of the work, including the equivalents of the results when expressed in modern notation, seem adequate and satisfactory. A number of geometrical proofs are also provided in certain cases where Barrow has omitted the proof of an important proposition.

T. L. HEATH.

**Leçons sur les Méthodes de Sturm dans la théorie des équations différentielles linéaires et leurs développements modernes, professées à la Sorbonne en 1913-1914 par Maxime Bôcher. Recueillies et rédigées par GASTON JULIA. Pp. vi, 118. 5 frs. 1917. (Gauthier-Villars et Cie.)**

The question of the determination of a function by boundary conditions in addition to satisfying a certain differential equation first arose in the study of the partial differential equations of mathematical physics, and it is enough

here to refer on this point to the names of Fourier, Green and Riemann. A method in use with such equations consists in the formation of products which satisfy this equation and whose factors each contain one variable only. Each factor then satisfies an ordinary linear differential equation, and we get boundary problems with certain ordinary differential equations. It seems that d'Alembert (1763) was the first to consider such a boundary problem: in his treatment of the vibrations of a non-homogeneous cord he arrived at the equation

$$y'' = \lambda \cdot \phi(x) \cdot y,$$

where  $y''$  denotes the second total differential quotient of  $y$  with respect to  $x$  and  $\phi(x)$  is a given positive function; and he asked whether we can so determine the parameter  $\lambda$  that the equation admits of a solution which vanishes at  $a$  and  $b$  without vanishing identically. He found that such a determination is possible, but he did not inquire into the question of the uniqueness or otherwise of this determination. The next man to investigate such questions was Sturm (1833). Sturm was markedly influenced by Fourier in the tendencies which Fourier had towards both algebra and mathematical physics, and his fundamental researches, which form the starting-point of those treated in this book, were published in the first volume (1836) of Liouville's *Journal*. Of the later works perhaps the most important is that of Klein in 1881. Historical accounts of the theory have been given by the author of this book in the German *Encyklopädie* (II. A. 7 a) and the first volume of the *Proceedings of the Fifth International Congress at Cambridge in 1912*.

This is a most excellent text-book on Sturm's methods; the first of three chapters deals with the theorems of existence of solutions of ordinary linear differential equations of the second order, the analogies between linear differential systems and linear algebraic systems, and real solutions and their zeros in the simplest cases. It is in this third chapter that Sturm's main theorems are treated: the first (p. 46) is that the zeros of the independent real solutions of a linear and homogeneous equation of the second order with real coefficients are mutually separated; the others are given on pp. 59, 60, 64. In the fourth chapter are treated two typical problems which surpass the problems of Sturm studied in the third chapter, and it is entitled "The Characteristic Functions and their Zeros in some more general cases"; and the last chapter is on those properties of Green's function which can be extended to ordinary linear differential equations.

PHILIP E. B. JOURDAIN.

**The Gyro Compass.** By LT.-COMMANDER G. B. HARRISON, of the Compass Department, Admiralty. 9d. 1917. (Published by the Stationery Office.)

Mathematics have been intentionally omitted. No notice is required then in the *Mathematical Gazette*.

But the opportunity may be taken of drawing attention to a curious quality of our official treatise, in the disregard of the rules of scientific politeness and urbanity.

The author is forbidden By Order to award any praise, for fear of raising a claim for reward or compensation, but he is encouraged to crab and disparage all innovations likely to disturb the peace of routine.

The result has been seen in the timid backward state of science in our military preparation and initiative.

The Gyro Compass was foreshadowed by Sir W. Thomson at the British Association in Montreal, 1884, "A Gyrostatic Model of the Mariner's Compass."

But the mechanical skill of Anschütz, and 25 years, was required to make the idea work.

So a word of praise might have been spared for these pioneers, to counteract subsequent carping criticism.

Any explanation in this tract does not extend beyond the Kindergarten stage, to say that one motion sets up another, up or down, right or left, but not venturing so far as to say how much.

A despondent feeling is left on us of the low state of our official theoretical science, when the author does not dare go into calculation or figures, for fear of scaring away all his readers.

C. V.

## CORRESPONDENCE.

TO THE EDITOR OF THE *Mathematical Gazette*.

UNIVERSITY COLLEGE, LONDON, W.C.

8th May, 1917.

DEAR SIR,

In my letter on p. 15 of your issue of January, 1917, I asked Prof. Lodge to explain how Rule 1, viz. :

"Multiplications and Divisions must be performed before Additions and Subtractions,"

can stand if Rule 2, viz. :

"Multiplications and Divisions must be performed in order (from left to right)," is abandoned.

The only part of Prof. Lodge's reply on p. 63 of your issue of March, 1917, which is material to my question, is the following :

"With regard to the term quoted by Prof. Hill, viz. :  $6 \div 3 \times 2$ , my own feeling is that it ought not to be written without brackets, viz. : that, if it means  $(6 \div 3) \times 2$ , which is in accordance with Rule 2, it ought to have been written in the unambiguous form  $(6 \times 2) \div 3$ ."

But this is only the introduction of a new rule or convention declaring the inadmissibility of expressions of the form  $a \div b \times c$ , in place of Rule 2. Is there also to be a convention regarding expressions of the form  $a \div b \div c$  ?

If I understand Prof. Lodge rightly he would reject this also, and consequently declare as inadmissible both the expressions

$$9 - 6 \div 3 \times 2 + 4 \quad \text{and} \quad 9 - 6 \div 3 \div 2 + 4.$$

I have never met with any convention of this kind before. On the other hand many writers admit them both and interpret them by the aid of Rules 1 and 2. As a matter of mere logic I can understand their procedure, though I regard it as harmful to the beginner. What I do not understand is Prof. Lodge's proposal to maintain Rule 1 but *nominally* to reject Rule 2. At the same time the words quoted above show that he in fact excludes from the category of admissible expressions a class of expression which he does not accurately define, and of which he gives only one example ; and this seems to me much more objectionable than the frank adoption of Rule 2. M. J. M. HILL.

## THE LIBRARY.

## CHANGE OF ADDRESS.

THE Library is now at 9 Brunswick Square, W.C., the new premises of the Teachers' Guild.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

## SCARCER BACK NUMBERS.

Reserves are kept of A.I.G.T. Reports and Gazettes, and, from time to time, orders come for sets of these. We are now unable to fulfil such orders for want of certain back numbers, which the Librarian will be glad to buy from any member who can spare them, or to exchange other back numbers for them :

Gazette No. 8 (very important).

A.I.G.T. Report No. 11 (very important).

A.I.G.T. Reports, Nos. 10, 12.

## ERRATA.

p. 66, last line but one, and l. 12, p. 68,

for  $-3px$  read  $+3px$ .

